MATH 25A – PRACTICE EXAM #2

(1) Let $A \in Mat(n,n)$. Prove that A satisfies a polynomial equation
$$A^k + a_{k-1}A^{k-1} + \ldots + a_0 = 0$$
for some $k > 0$ and $a_0, \ldots, a_{k-1} \in \mathbb{R}$.

(2) Let $A \in Mat(m,n)$.
  (a) Prove that there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ such that the dot product $Av_i \cdot Av_j = 0$ for $i \neq j$. (Hint: consider eigenvectors of $A^T A$.)
  (b) A matrix $O \in Mat(n,n)$ is called orthogonal if its columns form an orthonormal basis of $\mathbb{R}^n$. Prove that there exist orthogonal matrices O_1 and O_2 such that
  $$B = O_1^T A O_2$$
  is a diagonal matrix: $B_{ij} = 0$ for $i \neq j$. (Assume for simplicity that $m = n$ and A defines a isomorphism $\mathbb{R}^n \to \mathbb{R}^n$. Can you do the general case?)

(3) Define $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2}^{a_n}$ for $n \geq 1$. Prove that the sequence a_n converges to a limit $a \leq 2$.

(4) Recall that the closure $\overline{S}$ of a set $S \in \mathbb{R}^n$ is the intersection of all closed sets containing S, or equivalently, the set of limits of convergent sequences in S (convergent in $\mathbb{R}^n$, not necessarily in S). Prove that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $f(\overline{B}) \subset \overline{f}(B)$ for any set $B \subset \mathbb{R}^n$.

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}.$$
  (a) Prove that Df is invertible at every point, hence f has a local inverse near every point.
  (b) Show that f does not have a global inverse.

(6) Find local maxima and minima of $x_1^3 + 3x_1x_2^2 - 3x_1^2 - 3x_2^2 + 4$.

(7) Assume the following fact: If $B \in Mat(n,n)$ is such that $|B| < 1$ then the sequence
$$I_n + B + B^2 + B^3 + \ldots$$
converges. In other words, the infinite sum is a well-defined matrix.
  (a) Prove that the infinite sum above is the inverse of $I_n - B$. (This is the matrix version of the formula
  $$\frac{1}{1-x} = 1 + x + x^2 + \ldots)$$

(b) Prove that the set of invertible matrices is open in $Mat(n,n) = \mathbb{R}^{n^2}$.
(c) Prove that the derivative of the map f sending a matrix $A \in Mat(n,n)$ to its inverse A^{-1} is

$$Df(A)(H) = -A^{-1}HA^{-1}.$$

(This corresponds to the formula

$$(\frac{1}{x})' = -\frac{1}{x^2})$$

(8) Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if and only if it is continuously differentiable, or find a counterexample to this claim.

(9) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that $f(x)$ approaches 0 as $|x|$ goes to infinity.
(a) Give an exact definition of "$f(x)$ approaches 0 as $|x|$ goes to infinity".
(b) Prove that there exists a point $a \in \mathbb{R}^2$ such that $Df(a) = 0$.

(10) For this problem, it is helpful to think of matrices as maps. Prove:
(a) If $rank(A)$ is maximal possible then A has a right inverse B:

$$AB = I.$$

(b) If $nullity(A) = 0$ then A has a left inverse C:

$$CA = I.$$