MATH 25A – PRACTICE EXAM #2

(1) Let $A \in Mat(n,n)$. Prove that A satisfies a polynomial equation

$$A^k + a_{k-1}A^{k-1} + \ldots + a_0 = 0$$

for some $k > 0$ and $a_0, \ldots, a_{k-1} \in \mathbb{R}$.

Proof. The matrices A^i are elements of a finite dimensional vector space, hence there can be only finitely many of them linearly independent.

(2) Let $A \in Mat(m,n)$.

(a) Prove that there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ such that the dot product $Av_i \cdot Av_j = 0$ for $i \neq j$. (Hint: consider eigenvectors of A^TA.)

(b) A matrix $O \in Mat(n,n)$ is called orthogonal if its columns form an orthonormal basis of $\mathbb{R}^n$. Prove that there exist orthogonal matrices O_1 and O_2 such that

$$B = O_1^T A O_2$$

is a diagonal matrix: $B_{ij} = 0$ for $i \neq j$. (Assume for simplicity that $m = n$ and A defines a isomorphism $\mathbb{R}^n \to \mathbb{R}^n$. Can you do the general case?)

Proof. Let O_2 be the matrix with columns consisting of the vectors v_i constructed in part (a). Then $O_2^T A^T A O_2$ is diagonal. Now set $O_1' = AO_2$. Then O_1' is almost orthogonal: by part (a) its columns are pairwise orthogonal. Assuming that A is an isomorphism, the columns of O_1' must be nonzero, so we can rescale them to unit length to obtain O_1.

(3) Define $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2}^{a_n}$ for $n \geq 1$. Prove that the sequence a_n converges to a limit $a \leq 2$.

Proof. Show that a_i is an increasing sequence, bounded from above by 2.

(4) Recall that the closure $\overline{S}$ of a set $S \in \mathbb{R}^n$ is the intersection of all closed sets containing S, or equivalently, the set of limits of convergent sequences in S (convergent in $\mathbb{R}^n$, not necessarily in S). Prove that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $f(\overline{B}) \subset \overline{f(B)}$ for any set $B \subset \mathbb{R}^n$.

Proof. First assume that f is continuous, $B \subset \mathbb{R}^n$ any set, and $y \in f(\overline{B})$. We have to show that $y \in \overline{f(B)}$. By definition, $y = f(b)$ such that b is the limit of some convergent sequence b_i in B. By continuity of f, the sequence $f(b_i)$ lies in $f(B)$ and converges to $f(b) = y$. Hence $y \in \overline{f(B)}$.

Conversely, suppose that f is not continuous at some $x \in \mathbb{R}^n$. Use the $\varepsilon - \delta$ definition to construct a sequence x_i converging to x such that $f(x_i)$ does not converge to $f(x)$; now take $B = \{x_1, x_2, \ldots\}$.

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix}.$$

(a) Prove that Df is invertible at every point, hence f has a local inverse near every point.
(b) Show that f does not have a global inverse.

Proof. For (b) show that f is not one-to-one.

(6) Find local maxima and minima of $x_1^3 + 3x_1x_2^2 - 3x_1^2 - 3x_2^2 + 4$.

Proof. The question here is, which ones of the critical points are local minima or maxima. Find the Hessian matrix at each such point and compute its signature.

(7) Assume the following fact: If $B \in Mat(n,n)$ is such that $|B| < 1$ then the sequence

$$I_n + B + B^2 + B^3 + \dots$$

converges. In other words, the infinite sum is a well-defined matrix.

(a) Prove that the infinite sum above is the inverse of $I_n - B$. (This is the matrix version of the formula

$$\frac{1}{1-x} = 1 + x + x^2 + \dots)$$

(b) Prove that the set of invertible matrices is open in $Mat(n,n) = \mathbb{R}^{n^2}$.
(c) Prove that the derivative of the map f sending a matrix $A \in Mat(n,n)$ to its inverse A^{-1} is

$$Df(A)(H) = -A^{-1}HA^{-1}.$$

(This corresponds to the formula

$$(\frac{1}{x})' = -\frac{1}{x^2})$$

Proof. For part (b) we have to show that there exists $\varepsilon > 0$ such that if A is invertible then $A + H$ is also invertible for $|H| < \varepsilon$. Part (a) proves this for $A = I_n$ and $\varepsilon = 1$. In general, show that $A + H$ multiplied with A^{-1} is invertible for $|H| < \varepsilon$, and deduce from this that $A + H$ is invertible for the same H.

(8) Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if and only if it is continuously differentiable.

Example. Consider the function $x^2 \sin \frac{1}{x}$.

(9) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that $f(x)$ approaches 0 as $|x|$ goes to infinity.

(a) Give an exact definition of "$f(x)$ approaches 0 as $|x|$ goes to infinity".
(b) Prove that there exists a point $a \in \mathbb{R}^2$ such that $Df(a) = 0$.

Definition We say that $\lim_{|x|\to\infty} f(x) = L$ if for any $\varepsilon > 0$ there exists M such that $|f(x) - L| < \varepsilon$ whenever $|x| > M$.

The proof of part (b) is similar to the proof in the other practice exam.

(10) For this problem, it is helpful to think of matrices as maps. Prove:

(a) If $rank(A)$ is maximal possible then A has a right inverse B:

$$AB = I.$$

Proof. Since $rank(A)$ is maximal possible, the linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ is surjective. For each standard basis vector $e_i \in \mathbb{R}^m$ we can choose $v_i \in \mathbb{R}^n$ such that $Av_i = e_i$. Now let v_i be the columns of B.

(b) If $nullity(A) = 0$ then A has a left inverse C:

$$CA = I.$$

Proof. If $nullity(A) = 0$ then $A : \mathbb{R}^n \to \mathbb{R}^m$ is injective, hence A maps $\mathbb{R}^n$ isomorphically onto its image. To construct C, we can invert $A : \mathbb{R}^n \to Img(A)$ and map the rest of $\mathbb{R}^m$ to anything in $\mathbb{R}^n$. More precisely, we know that the set $\{Ae_1, \dots, Ae_n\}$ is linearly independent. Extend this to a basis of $\mathbb{R}^m$, $\{v_1 = Ae_1, \dots, v_n = Ae_n, v_{n+1}, \dots, v_m\}$. Now let $L : \mathbb{R}^m \to \mathbb{R}^n$ be the linear map defined on the basis: $L(v_i) = e_i$ for $i = 1, \dots, n$, and $L(v_i) = 0$ for $i = n+1, \dots, m$. Finally let C be the matrix of L.