MATH 25A – PROBLEM SET #10
DUE FRIDAY DECEMBER 7

1. Part A

(1) Let $C \subset \mathbb{R}^2$ be a smooth curve defined by $F(x,y) = 0$. Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a differentiable map such that the image of ϕ lies in C. Prove that for any $t \in \mathbb{R}$ we have

$$Img(D\phi(t)) \subset T_{\phi(t)}C,$$

and that the equality holds if $D\phi(t) \neq 0$.

(2) Generalize the previous problem to the case of a smooth k-dimensional manifold $M \subset \mathbb{R}^n$ and a differentiable map $\phi : \mathbb{R}^m \to M \subset \mathbb{R}^n$. If $m = k$, when is $T_{\phi(x)}M$ equal to the image of $D\phi(x)$?

(3) Let $S \subset \mathbb{R}^3$ be a smooth surface, and $f : S \to \mathbb{R}$ a function. We want to define when f is differentiable at some point $a \in S$, but we cannot do it directly because f is defined only on S, so partial derivatives of f may not exist. The differentiability of f is defined as follows. Near a point $a \in S$ we can write one coordinate, say z, as a function of the other two: $z = g(x,y)$, hence we get a function of two variables:

$$\overline{f}(x,y) = f(x,y,g(x,y)).$$

We say that f is differentiable at $a = (a_1, a_2, a_3)$ if $\overline{f}$ is differentiable at (a_1, a_2).

(a) If $h : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, show that the restriction $f = h|_S$ is differentiable.

(b) Note that if $a \in S$ is a local maximum of a differentiable function $f : S \to \mathbb{R}$, then (a_1, a_2) is a local maximum of $\overline{f} : \mathbb{R}^2 \to \mathbb{R}$, hence both partial derivatives of $\overline{f}$ must vanish at (a_1, a_2). Assume that $f = h|_S$ for some differentiable function $h : \mathbb{R}^3 \to \mathbb{R}$, and f has a local maximum at a. Prove that $grad(h)(a)$ is perpendicular to the surface S at a (that means, it is perpendicular to the tangent plane to the surface at a, or it is zero).

2. Part B

(1) Problem A.9.3.
(2) Prove Corollary A.9.3.
(3) Problem 3.3.13.

3. Part C

(1) Problem 3.5.1. In part (d) you have to find a matrix A such that for $p(x) = a_k x^k + \ldots + a_0$ and $q(x) = b_k x^k + \ldots + b_0$, we have $B(p,q) = \vec{a}^t A \vec{b}$.
(2) Problem 3.5.2.
(3) Problem 3.7.6.