Problem Set 10, Part B Solution Set

Tony Várilly

Math 25a, Fall 2001

1. Problem A9.3.

   Prove that if g is a $(k+1)$-times continuously differentiable on $(a-R, a+R)$, then, for $|h| < R$,

   $$g(a + h) = g(a) + g'(a)h + \cdots + \frac{1}{k!}g^{(k)}h^k + \frac{1}{k!}\int_0^h (h - t)^k g^{k+1}(a + t)\, dt. \qquad (1)$$

   Use induction and integration by parts.

   Solution. For $k = 0$, (1) reduces to

   $$g(a + h) = g(a) + \int_0^h g'(a + t)\, dt,$$

   which is true by the Fundamental Theorem of Calculus. Suppose that the result holds for a certain value of k and that g is C^{k+2}. Using integration by parts,

   $$\frac{1}{k!}\int_0^h (h - t)^k g^{k+1}(a + t)\, dt = -\frac{1}{(k + 1)!}(h - t)^{k+1} g^{(k+1)}(a + t)\Big|_0^h$$
   $$+ \frac{1}{(k + 1)!}\int_0^h (h - t)^{k+1} g^{(k+2)}(a + t)\, dt. \qquad (2)$$

   Substituting (**??**) in (1) we obtain

   $$g(a + h) = g(a) + g'(a)h + \cdots + \frac{1}{(k + 1)!}g^{(k+1)}h^{k+1} + \frac{1}{(k + 1)!}\int_0^h (h - t)^{k+1} g^{k+2}(a + t)\, dt.$$

   and this completes the induction. $\qquad\qquad\square$

2. Prove Corollary A9.3.

   If $|f^{k+1}(a + t)| \leq C$ for $0 \leq t \leq h$, then

   $$|f(a + h) - P_{f,a}^k(a + h)| \leq \frac{C}{(k + 1)!}h^{k+1}. \qquad (3)$$

Solution. From problem 1, it follows that

$$|f(a+h) - P_{f,a}^k(a+h)| = \left| \frac{1}{k!} \int_0^h (h-t)^k f^{k+1}(a+t)\, dt \right|.$$

Since $|f^{k+1}(a+t)| \leq C$ for $0 \leq t \leq h$,

$$\left| \frac{1}{k!} \int_0^h (h-t)^k f^{k+1}(a+t)\, dt \right| \leq \frac{1}{k!} \int_0^h \left| (h-t)^k f^{k+1}(a+t) \right| dt$$
$$\leq C \frac{1}{k!} \int_0^h \left| (h-t)^k \right| dt$$
$$= \frac{C}{(k+1)!} h^{k+1}.$$

$\square$

3. **Problem 3.3.13**

   (a) Let $f(x) = e^x$, so that $f(1) = e$. Show that

   $$e = \sum_{i=0}^k \frac{1}{i!} + r_{k+1} \quad \text{where} \quad |r_{k+1}| \leq \frac{3}{(k+1)!}. \tag{4}$$

   Solution. For all $t \in [0,1]$, one has $|f^{(k+1)}(t)| = e^t \leq e < 3$. It follows, using the result from problem 2, that

   $$|f(1) - P_{f,0}^k(1)| = \left| e - \sum_{i=0}^k \frac{1}{i!} \right| \leq \frac{3}{(k+1)!}.$$

   Then let $r_{k+1} = e - \sum_{i=0}^k 1/i!$. Then (4) follows immediately. $\square$

   (b) Prove that e is irrational: if $e = a/b$ for some integers a and b, deduce from part (a) that

   $$|k!a - bm| \leq \frac{3b}{k+1}, \quad \text{where} \quad m = \frac{k!}{0!} + \frac{k!}{1!} + \cdots + \frac{k!}{k!}. \tag{5}$$

   Conclude that if k is large enough then $k!a - bm$ is an integer that is arbitrarily small, and therefore 0.

   Solution. From part (a),

   $$\left| \frac{a}{b} - \sum_{i=0}^k \frac{1}{i!} \right| \leq \frac{3}{(k+1)!}$$

   Now multiply by $bk!$ to get (5). For any k such that $k > 3b - 1$ we get

   $$0 \leq |k!a - bm| < 1$$

   and since $|k!a - bm|$ is an integer, $k!a = bm$ for such values of k. $\square$

(c) Finally, observe that k does not divide m evenly, since it does divide every summand except the last one. Since k may be freely chosen, provided only that it is sufficiently large, take k to be a prime number larger than b. Then in $k!a = bm$ we have that k divides the left side, but it does not divide m. What conclusion do you reach?

Solution. What to do... What to do... Is there anything left for me to say? Am I missing something here? I guess I could repeat that entire last paragraph. Or maybe I could say: "We conclude that e must be irrational." I think that'll do. □