Problem Set #10A − Solutions

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Problem 1.

Claim 1.1. Let $C \subset \mathbb{R}^2$ be a smooth curve defined by $F(x, y) = 0$. Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be a differentiable map such that the image of ϕ lies in C. Then for any $t \in \mathbb{R}$,

$$\mathrm{Img}(D\phi(t)) \subset T_{\phi(t)}C$$

and that these two sets are equal if $D\phi(t) \neq 0$.

This is a special case of problem two. (One must recall, of course, that if the derivative of a map $\mathbb{R}^2 \to \mathbb{R}$ is nonzero, then it has maximal rank.)

$\square$

Problem 2.

Claim 2.1. Let $M \subset \mathbb{R}^n$ be a smooth k-manifold defined as the kernel of $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$, and let $\phi : \mathbb{R}^m \to M$ be differentiable. Then

$$\mathrm{Img}(D\phi(t)) \subset T_{\phi(x)}M$$

and, when $m = k$, these two sets are equal if $D\phi(t)$ has maximal rank.

Because M is smooth, F is differentiable. But, because $\mathrm{Img}(f) \subset M$, we know that $F \circ \phi$ is everywhere zero. We thus have, for all $t \in \mathbb{R}^m$:

$$0 = D(F \circ \phi)(t) = DF(\phi(t)) \cdot D\phi(t)$$

Therefore, for all $v \in \mathrm{Img}(D\phi)$, we know $DF(\phi(t)) \cdot v = 0$, and thus $v \in \ker(DF(\phi(t)) = T_{\phi(t)}M$. Consequently,

$$\mathrm{Img}(D\phi) \subset T_{\phi(t)}M$$

Now, suppose that $m = k$ and $D\phi(t)$ has maximal rank (i.e. rank k). Then, we know that $\dim(\mathrm{Img}(D\phi(t))) = k$. Conversely, because M is smooth, DF has rank maximal rank (i.e. rank $n - k$), giving $\dim T_{\phi(t)}M = k$. Finally, we note the both $\mathrm{Img}(D\phi(t)))$ and $T_{\phi(t)}M$ are subspaces of $\mathbb{R}^n$ and thus $\mathrm{Img}(D\phi(t))$ is a subspace of $T_{\phi(t)}M$. Because they both have the same dimension, we can conclude that

$$\mathrm{Img}(D\phi(t))) = T_{\phi(t)}M$$

$\square$

Problem 3.

Let $S \subset \mathbb{R}^3$ be a smooth surface.

Part i.

Claim 3.1. Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. Then the restriction $f := h \mid_S$ is differentiable.

Let $a \in S$. Then, wlog, S can be expressed, around a, as $\left\{ \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$ for some smooth $g : \mathbb{R}^2 \to \mathbb{R}$. We will define $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix}$$

We know that f is differentiable iff $\overline{f} := f \circ \phi$ is. Well, we know that $f = h \mid_S$, thus $\overline{f} = (h \mid_S) \circ \phi = h \circ \phi$. We already know that h is differentiable, so if we can prove that ϕ is as well, then $\overline{f}$ is the composition of two differentiable function and is thus differentiable itself. To do this, we calcluate

$$\mathfrak{J}\phi(a_1, a_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ D_1 g(a_1, a_2) & D_2 g(a_1, a_2) \end{bmatrix}$$

Because all of the partial derivatives are well-defined and continuous, $\mathfrak{J}\phi = D\phi$, proving that ϕ is differrntiable.

$\square$

Part ii.

Let $a \in S$, and let the third coordinate of points on S near a be given implicitly as a smooth function, $g : \mathbb{R}^2 \to \mathbb{R}$, of the first two.

Claim 3.2. Let $f : S \to \mathbb{R}$ be a differentiable function such that $f = h \mid_S$ for some differentiable function $h : \mathbb{R}^3 \to \mathbb{R}$ and such that f has a local maximum at a. Then $\vec{\nabla} h(a)$ is perpendicular to S at a. That is, either $\vec{\nabla} h(a) = 0$ or $\vec{\nabla} h(a)$ is perpendicular to $T_a S$.

Because S is a smooth manifold, there exists a function $F : \mathbb{R}^3 \to \mathbb{R}$ such that $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid F(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = 0 \right\}$. We define ϕ as in the preceding claim. We then see that

$$D\phi(a_1, a_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ D_1 g(a_1, a_2) & D_2 g(a_1, a_2) \end{bmatrix}$$

where $a =: (a_1, a_2, a_3)$. This clearly has rank 2, thus by question 2, we know that

$$\text{Img}(D\phi(a_1, a_2)) = T_a S$$

Now, let $v \in \text{Img}(D\phi(a_1, a_2))$. Then $v = D\phi(a_1, a_2)(u)$ for some u. Consider

$$\begin{aligned} \vec{\nabla} h(a) \cdot v &= (Df(a_1, a_2, a_3))(v) \\ &= (Df(a_1, a_2, a_3))(D\phi(a_1, a_2))(u) \\ &= D(f \circ \phi)(a_1, a_2)(u) \\ &= D\overline{f}(a_1, a_2)(u) \\ &= 0 \end{aligned}$$

since $\overline{f}$ is differentiable (by the previous part) and since the local maximum of f at a implies that $D\overline{f}(a_1, a_2) = 0$. Thus $\forall v \in \text{Img}(D\phi(a_1, a_2)) = T_a S$ we have $\vec{\nabla}(a) \cdot v = 0$ and thus $\vec{\nabla}(a)$ is perpendicular to $T_a S$.

$\square$