Problem Set #1 – Solutions

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Math 25

Problem 1.

Let $f : S \to T$ be a map of sets, $A, B \subset S$ and $C, D \subset T$.

Part i.

Claim 1.1. $f(A \cap B) \subset f(A) \cap f(B)$

We will show $y \in f(A \cap B) \Rightarrow y \in f(A) \cap f(B)$. Let $y \in f(A \cap B)$ be given. Then $\exists x \in A \cap B$ such that $f(x) = y$. Then $x \in A$ and $x \in B$, so $y \in f(A)$ and $y \in f(B)$. Consequently, $y \in f(A) \cap f(B)$.

□

Part ii.

Claim 1.2. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

We will prove this in two steps.

Step 1: $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$**.** Let $x \in f^{-1}(C \cap D)$ be given. Then $f(x) \in f(C)$ and $f(x) \in f(D)$. Thus $y \in f^{-1}(C)$ and $y \in f^{-1}(D)$, meaning $y \in f^{-1}(C) \cap f^{-1}(D)$.

Step 2: $f^{-1}(C \cap D) \supset f^{-1}(C) \cap f^{-1}(D)$**.** Let $x \in f^{-1}(C) \cap f^{-1}(D)$ be given. Then $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. Thus $\exists y \in C$ and $z \in D$ such that $x \in f^{-1}(y)$ and $x \in f^{-1}(z)$. But then $y = f(x) = z$, so $y \in C \cap D$, making $x \in f^{-1}(C \cap D)$.

Putting these two steps together yields the claim.

□

Problem 2.

Claim 2.1. Let $A, B \subset S$. Then $(A \subset B) \Leftrightarrow (S - B \subset S - A)$.

$\Rightarrow$. (Assume $A \subset B$ in order to prove $S - B \subset S - A$.) We must show $x \in S - B \Rightarrow x \in S - A$, so let $x \in S - B$ be given. Then $x \notin B$, so because $A \subset B$, $x \notin A$ either. Because $x \in S$ but $x \notin A$, $x \in S - A$.

$\Leftarrow$. (Assume $S - B \subset S - A$ in order to prove $A \subset B$.) We must show $x \in A \Rightarrow x \in B$, so let $x \in A$ be given. Then $x \notin S - A$. We now recall the contrapositive of our assumption:

$$y \notin S - A \Rightarrow y \notin S - B$$

Thus $x \notin S - B$. But because $x \in A \subset S$ (so $x \in S$), $x \in B$.

□

Problem 3.

Claim 3.1. Statements (a), (b), and (c) are equivalent.

We will prove the claim in several steps:

(a) ⇒ (b): We assume f is an injection in order to prove $\left|f^{-1}(t)\right| \leq 1$ for all t. Let t be given. If $f^{-1}(t) = \varnothing$, then (b) is satisfied immediately. Otherwise, let $s_1, s_2 \in f^{-1}(t)$. Then $f(s_1) = f(s_2)$, so by the injectivity of f, $s_1 = s_2$ guaranteeing $\left|f^{-1}(t)\right| = 1$.

(b) ⇒ (c): We assume $\forall t, \left|f^{-1}(t)\right| \leq 1$ in order to prove $f(A \cap B) = f(A) \cap f(B)$. We recall from Problem 1, that $f(A \cap B) \subset f(A) \cap f(B)$. Thus it suffices to prove that $f(A) \cap f(B) \subset f(A \cap B)$. Let $x \in f(A) \cap f(B)$ be given. Then $\exists a \in A$ such that $f(a) = x$ and $\exists b \in B$ such that $f(b) = x$. Therefore, $a, b \in f^{-1}(x)$, so by assumption $a = b$. Thus, $a \in A \cap B$, so $x = f(a) \in f(A \cap B)$.

(c) ⇒ (a): We assume $f(A \cap B) = f(A) \cap f(B)$ in order to prove that f is injective. Let s_1, s_2 be given such that $f(s_1) = f(s_2)$. Define $A = \{s_1\}$ and $B = \{s_2\}$. Then $f(s_1) \in f(A) \cap f(B)$. Therefore, $f(s_1) \in f(A \cap B)$, requiring that $A \cap B \neq \varnothing$. Therefore, $A \cap B \ni s_1 = s_2$, and the claim is proven.

□

Problem 4.

In both parts, we proceed by induction.

Part i.

Claim 4.1. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Base Case:

$$\begin{aligned} \sum_{i=1}^{1} i^3 &= \frac{1^2(1+1)^2}{4} \\ 1 &= 1 \end{aligned}$$

Inductive Step: Assume the claim is true when $n = k$ in order to prove that the claim is true when $n = k + 1$. We take

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \end{aligned}$$

(Several lines of algebra omitted)

$$= \frac{(k+1)^2\left((k+1)+1\right)^2}{4}$$

□

Part ii.

Claim 4.2. $\sum_{i=1}^{n} 3^{2i-1} = \frac{3(9^n-1)}{8}$

Base Case:

$$\begin{aligned} \sum_{i=1}^{1} 3^{2i-1} &= \frac{3(9^1-1)}{8} \\ 3 & \quad 3 \end{aligned}$$

Inductive Step: Assume the claim is true when $n = k$ in order to prove that the claim is true when $n = k + 1$. We take

$$\begin{aligned}
\sum_{i=1}^{k+1} 3^{2i-1} &= \sum i = 1^k 3^{2i-1} + 3^{2(k+1)-1} \\
&= \frac{3(9^k - 1)}{8} + 3^{2(k+1)-1} \\
\text{(Several lines of algebra omitted)} & \\
&= \frac{3(9^{n+1} - 1)}{8}
\end{aligned}$$

□

Problem 5.

Claim 5.1. The relation $m \sim n$ iff $12 \mid m - n$ is an equivalence relation.

We must check three axioms:

Reflexivity: ($m \sim m$.**)** Let $m \in \mathbb{Z}$ be given. Then

$$12 \mid 0 = m - m$$

so $m \sim m$.

Symmetry: ($(m \sim n) \Rightarrow (n \sim m)$.**)** Let $m, n \in \mathbb{Z}$ be given such that $m \sim n$. Thus $12 \mid m - n$. Then

$$12 \mid -(m - n) = n - m$$

so $n \sim m$.

Transitivity: ($(m \sim n, n \sim p) \Rightarrow (m \sim p)$.**)** Let $m, n, p \in \mathbb{Z}$ be given such that $m \sim n$ and $n \sim p$. Then $12 \mid m - n$ and $12 \mid n - p$. Thus

$$12 \mid (m - n) + (n - p) = (m - p)$$

making $m \sim p$.

□

Problem 6.

Claim 6.1. Let $\bar{a}, \bar{b}$ be two equivalence classes under $\sim$. Then either $\bar{a} = \bar{b}$ (as sets), or $\bar{a} \cap \bar{b} = \varnothing$.

Let $\bar{a}, \bar{b}$ be given. There are two cases:

Case 1: $\bar{a} \cap \bar{b} = \varnothing$**.** In this case, the claim is satisfied immediately.

Case 2: $\bar{a} \cap \bar{b} \neq \varnothing$**.** Let $x \in \bar{a} \cap \bar{b}$. Then, by definition, $x \sim a$ and $x \sim b$, so by transitivity, $a \sim b$. Now, let $a' \in \bar{a}$. Then $a' \sim a$, so by transitivity, $a' \sim b$ and $a' \in \bar{b}$. Thus $\bar{a} \subset \bar{b}$. Conversely, let $b' \in b$. Then $b' \sim b$, so by transivity, $b' \sim a$ and $b' \in \bar{a}$. Thus $\bar{b} \subset \bar{a}$. Therefore $\bar{a} = \bar{b}$ (as sets), and the claim is proven.

□

Problem 7.

Part i.

Claim 7.1. The relation $(a, b) \sim (c, d)$ iff $ad = bc$ is an equivalence relation.

We must check three axioms:

Reflexivity: ($x \sim x$.) Let $(a, b) \in \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ be given. Then $a \cdot b = b \cdot a$, making $(a, b) \sim (a, b)$.

Symmetry: ($(x \sim y) \Rightarrow (y \sim x)$.) Let $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ be given such that $(a, b) \sim (c, d)$. Then $a \cdot d = b \cdot c \Rightarrow c \cdot b = d \cdot a$, making $(c, d) \sim (a, b)$.

Transitivity: ($(x \sim y, y \sim z) \Rightarrow (x \sim z)$.) Let $(a, b), (c, d), (e, f) \in \mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ be given such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$. Consequently,

$$af = \frac{ad}{d} \cdot \frac{cf}{c} = \frac{bc}{d} \cdot dec = be$$

making $(a, b) \sim (e, f)$

□

Part ii.

Because the second term of the ordered pair is nonzero, we can define a surjection $f : \mathbb{Z} \times (\mathbb{Z} \backslash \{0\}) \to \mathbb{Q}$ given by

$$(a, b) \mapsto \frac{a}{b}$$

(This is a surjection, because any rational number can be written as $\frac{a}{b}$ for some $a, b \in \mathbb{Z}$ where $b \neq 0$. Thus, $(a, b) \in f^{-1}\left(\frac{a}{b}\right) \neq \varnothing$). Furthermore, we will find that $(a, b) \sim (c, d)$ iff $f((a, b)) = f((c, d))$. *Thus we have a bijection between equivalence classes of $\mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ under $\sim$ and the set of rational numbers, $\mathbb{Q}$.*

Problem 8.

Part i.

Claim 8.1. The relation defined by $s_1 \sim s_2$ iff $f(s_1) = f(s_2)$ is an equivalence relation.

We must check three axioms:

Reflexivity: Let $s \in S$ be given. Then $f(s) = f(s)$ so $s \sim s$.

Symmetry: Let $s_1, s_2 \in S$ be given such that $s_1 \sim s_2$. Then $f(s_1) = f(s_2)$ so $f(s_2) = f(s_1)$ making $s_2 \sim s_1$.

Transitivity: Let $s_1, s_2, s_3 \in S$ be given such that $s_1 \sim s_2$ and $s_2 \sim s_3$. Then $f(s_1) = f(s_2)$ and $f(s_2) = f(s_3)$. Consequently, $f(s_1) = f(s_3)$, making $s_1 \sim s_3$.

□

Part ii.

Define $\bar{f} : S/\sim \to T$ by $\bar{a} \mapsto f(a)$.

Claim 8.2. f is an injective map.

We must first prove that f is well-defined. Specifically, let $a_1, a_2 \in \bar{a}$, then we must prove $f(a_1) = f(a_2)$. This follows immediately from the definition of $\sim$.

We must next prove that $\bar{f}$ is injective. Let $\bar{a}, \bar{b}$ be two equivalence classes under $\sim$ such that $\bar{f}(\bar{a}) = \bar{f}(\bar{b})$. Then $f(a) = f(b)$ making $a \sim b$. By problem 6, we have $\bar{a} = \bar{b}$ making the map $\bar{f}$ injective.

□