Problem Set 1, Part B Solution Set

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1. Let $\{a_i\}$ and $\{b_i\}$ be two sequences in $\mathbb{R}$.

   (a) Prove that if $\{a_i\}$ converges to a and $\{b_i\}$ converges to b then the sequence $\{a_i + b_i\}$ converges to $a + b$.

   Solution. Let $\epsilon > 0$ be given. Since $\{a_i\}$ converges to a, there exists N_1 such that $|a_i - a| < \epsilon/2$ whenever $i > N_1$. Similarly, there exists N_2 such that $|b_i - b| < \epsilon/2$ whenever $i > N_2$. Let $N = \max\{N_1, N_2\}$. then, by the triangle inequality,

   $$|(a_i + b_i) - (a + b)| = |(a_i - a) + (b_i - b)| \leq |a_i - a| + |b_i - b| < \epsilon/2 + \epsilon/2 = \epsilon,$$

   for $i > N$. Thus the sequence $\{a_i + b_i\}$ converges to $a + b$. $\square$

   (b) Prove that if $\{a_i\}$ converges to a and the sequence $\{a_i - b_i\}$ converges to 0, then $\{b_i\}$ also converges to a.

   Solution I. Very few people noticed that this problem follows directly from part (a). If $\{a_i - b_i\}$ converges to 0, so does the sequence $\{b_i - a_i\}$. Since $\{a_i\}$ converges to a, the sequence $\{a_i + b_i - a_i\} = \{b_i\}$ must converge to $a + 0 = a$ by part (a). $\square$

   Solution II. Given $\epsilon > 0$, there exists N_1 such that $|a_i - a| < \epsilon/2$ whenever $i > N_1$. Similarly, there exists N_2 such that $|b_i - a_i - 0| < \epsilon/2$ whenever $i > N_2$. Let $N = \max\{N_1, N_2\}$. then, by the triangle inequality,

   $$|b_i - a| = |(b_i - a_i) + (a_i - a)| \leq |b_i - a_i| + |a_i - a| < \epsilon/2 + \epsilon/2 = \epsilon,$$

   for $i > N$. Hence the sequence $\{b_i\}$ converges to a. $\square$

2. Let $\{a_i\}$ be a sequence in $\mathbb{R}$. Define a new sequence

   $$b_i = \frac{a_1 + \ldots + a_i}{i}.$$

   (a) Prove that if $\{a_i\}$ converges to a then $\{b_i\}$ also converges to a.

Solution. Let $\epsilon > 0$ be given. We know $\{a_i\}$ converges to a, so there is an N_1 such that $|a_i - a| < \epsilon/2$ whenever $i > N_1$. We have

$$\begin{aligned}|b_i - a| = \left|\frac{a_1 + \cdots + a_i}{i} - a\right| &= \left|\frac{(a_1 - a) + \cdots + (a_i - a)}{i}\right| \\ &= \left|\frac{(a_1 - a) + \cdots + (a_{N_1} - a)}{i} + \frac{(a_{N_1+1} - a) + \cdots + (a_i - a)}{i}\right| \\ &\le \frac{|a_1 - a| + \cdots + |a_{N_1} - a|}{i} + \frac{|a_{N_1+1} - a| + \cdots + |a_i - a|}{i}.\end{aligned}$$

Let $K = |a_1 - a| + \cdots + |a_{N_1} - a|$. Notice that once we have picked our ϵ, this quantity is fixed. We also know that $|a_n - a| < \epsilon/2$ for $n = N_1 + 1, \ldots, i$. Therefore

$$|b_i - a| < \frac{K}{i} + \frac{(i - N_1)}{i} \cdot \frac{\epsilon}{2}.$$

Since K is constant, pick N_2 large enough so that $K/N_2 < \epsilon/2$. Then, for $i > \max\{N_1, N_2\}$

$$|b_i - a| < \frac{K}{i} + \frac{(i - N_1)}{i} \cdot \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so $\{b_i\}$ converges to a. □

(b) find a sequence $\{a_i\}$ that does not converge, such that $\{b_i\}$ converges.

Solution. Many people skipped this question altogether. There are many examples of such sequences. A popular one was $a_i = (-1)^i$. This sequences oscillates between 1 and -1. But $\{b_i\} = \{-1, 0, -1/3, 0, -1/5, \ldots\}$, and this sequence does converge. Indeed, we have $|a_1 + \cdots + a_n| \le 1$, so

$$\left|\frac{a_1 + \cdots a_i}{i}\right| \le \frac{1}{i}.$$

Thus, for a given $\epsilon > 0$, we can choose $N = 1/\epsilon$ and $\{b_i\}$ converges to 0. □

Consider the two properties of real numbers:

C1. Every nonempty subset $X \subset \mathbb{R}$ which is bounded from above has a least upper bound.

C2. Every non-decreasing bounded from above sequence in $\mathbb{R}$ has a limit.

3. In class we proved that $C1$ implies $C2$. Prove the converse: $C2$ implies $C1$.

Remark. This exercise caused a lot problems. Many did not know how to start it off. We are trying to prove that a non-empty set X that is bounded above has a least upper bound. We are given that every non-decreasing sequence that is bounded above converges. So the appropriate strategy to attack the problem is as follows:

- Start with an arbitrary non-empty set X that is bounded above.
- Construct somehow a sequence, which *need not use all the elements of* X, that is non-decreasing and bounded above.

- Use $C2$ to claim this sequence converges.
- Prove that the limit of our sequence is a least upper bound for X (provided we were clever enough in our construction of the sequence.

Many people adopted this strategy but claimed that one could take *all* elements of X and put them into a non-decreasing sequence. This can't be done in general. For example, if $X = (0, 1)$, then X has an *uncountable* number of elements, which means that the elements of X cannot be put into a sequence which is indexed by the natural numbers.

Solution. This proof comes straight out of Hubbard & Hubbard, p 7–8. We basically transcribe it here for completeness' sake. Suppose $x \in X$ and a is an upper bound for X. Assume $x > 0$ (if $x \leq 0$ our proof must be modified a little). If $x = a$ then a is our least upper bound.

"If $x \neq a$, there is a first digit j such that the j^{th} digit of x is smaller than the j^{th} digit of a. Consider all the numbers in $[x, a]$ that can be written using only j digits after the decimal, than all zeroes. This is a finite non-empty set. In fact, it has at most 10 elements and $[a_j]$ is one of them[1]. Let b_i be the largest which is not an upper bound. Now consider the set of numbers in $[b_j, a]$ that have only $j+1$ digits after the decimal point, than all zeroes. Again this is a finite non-empty set, so you you can choose the largest which is not an upper bound; call it b_{j+1}. It should be clear that b_{j+1} is obtained by adding one digit to b_j. Keep going this way, defining numbers $b_{j+2}, b_{j+3}, \cdots$, each time adding one digit to the previous number."

We have created a non-decreasing sequence that is bounded above. By $C2$, it converges. Let b be the number whose k^{th} decimal is the same as that of b_k. The claim is that this number (which we secretly know is the limit of the sequence we constructed) is the least upper bound of X.

First, we show that b is an upper bound for x. If it wasn't there'd be a $y \in X$ such that $y > b$. This means there'd be a first k^{th} digit of y differing from the k^{th} digit of b. Then b_k would not be the largest number with k digits that is not an upper bound for X. This is a contradiction, so b must be an upper bound for X.

Now we show that b is the least upper bound for X. Suppose there is a b' that is also an upper bound for X and $b' < b$. There'd be a first k^{th} digit of b' differing from the k^{th} digit of b. Then b_k would be an upper bound for X. But this is a contradiction. □

4. Prove that if a sequence converges then it is a Cauchy sequence.

Solution. Say the sequence $\{a_i\}$ converges to a. This means that for a given $\epsilon > 0$ there exists N such that

$$|a_i - a| < \frac{\epsilon}{2} \quad \text{for all } i > N.$$

Let $i, j > N$. Then $|a_i - a| < \epsilon/2$ and $|a_j - a| < \epsilon/2$, and so, by the triangle inequality

$$|a_i - a_j| = |(a_i - a) + (a - a_j)| \leq |a_i - a| + |a - a_j| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This means $\{a_i\}$ is a Cauchy sequence. □

[1] $[a_j]$ denotes the finite decimal consisting of all the digits of a before the decimal, and j digits after the decimal

5. Prove that a Cauchy sequence is bounded (from above and below).

Solution. Let $\{a_i\}$ be a Cauchy sequence. Choose $\epsilon = 1$. There exists an N such that

$$|a_i - a_j| < 1 \quad \text{for all } i, j \geq N.$$

Now fix $j = N$ and add $|a_j|$ to both sides of the inequality above. Using the triangle inequality, we get $|a_i| < |a_N| + 1$ for all $i \geq N$. Let $B = \max\{|a_1|, |a_2|, \ldots, |a_N|\} + 1$. Then we have $|a_i| < B$ for all i. So our Cauchy sequence is bounded from both above and below. □

6. Let $\{a_i\}$ be a Cauchy sequence. Define a new sequence

$$b_i = \sup\{a_i, a_{i+1}, a_{i+2}, \ldots\}.$$

(a) Prove that the sequence $\{b_i\}$ converges to some number b.

Solution. Consider the statement:

$C2'$ Every non-increasing sequence that is bound below converges.

It is not hard to show the equivalence between $C2$ and $C2'$. We'll show that $\{b_i\}$ forms a non-increasing sequence that is bounded below. It will follow that the sequence converges to some number b. First, note that since $\{a_i, a_{i+1}, \ldots\} \subset \{a_{i+1}, a_{i+2}, \ldots\}$, it follows that $b_i \geq b_{i+1}$, with strict inequality if $b_i \neq a_i$. From problem 5, we know that $\{a_i\}$ is bounded $\Rightarrow$ there is some B such that $a_i > B$ for all i. But $b_i \geq a_i$ for all i, so $b_i > B$ for all i, that is, the sequence $\{b_i\}$ is bounded below. □

(b) Prove that the sequence $\{a_i\}$ also converges to the same number b.

Solution. We know $\{b_i\}$ converges to some number b. If we can show that $\{a_i - b_i\}$ converges to 0, then it will follow, by problem 1(b), that $\{a_i\}$ converges to b as well. Let $\epsilon > 0$ be given. Since $\{a_i\}$ is Cauchy, there exists N such that

$$|a_N - a_i| < \frac{\epsilon}{2} \quad \text{for all } i \geq N.$$

Since $b_j = \sup\{a_i, a_{i+1}, \ldots\}$ we have

$$|a_N - b_i| \leq \frac{\epsilon}{2} \quad \text{for all } i \geq N.$$

Now use the triangle inequality to get

$$|a_i - b_i| = |(a_i - a_N) + (a_N - b_i)| \leq |a_i - a_N| + |a_N - b_i| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for } i \geq N.$$

This gives the desired convergence of $\{a_i - b_i\}$. □

7. Let $C3$ be the property that every Cauchy sequence in $\mathbb{R}$ converges. Prove that $C3$ is equivalent to $C1$ (or to $C2$).

Solution. ($C2 \Rightarrow C3$) We wish to prove that Cauchy sequences converge. Given a Cauchy sequence $\{a_i\}$, we know it is bounded by problem 5, so by $C1$ (which we have already shown equivalent to $C2$), the *set* $\{a_i\}$ has a least upper bound (we have not established yet that the sequence converges). This means that our sequence $\{b_i\}$ from problem 6(a) exists. This sequence was shown to converge using $C2$, and in problem 6(b), we showed that $\{a_i\}$ converges to the same limit as $\{b_i\}$ does. Thus any Cauchy sequence is convergent.

($C3 \Rightarrow C2$) Let $\{a_i\}$ be a non-decreasing sequence that is bounded above by A. All the terms of the sequence are between a_1 and A. Divide the interval $[a_1, A]$ into subintervals of length $\epsilon > 0$. Then at least one of these intervals must contain infinitely many terms of the sequence. Otherwise, there would only be a finite number of terms in our sequence. Moreover, only *one* interval has infinitely many terms, because the sequence is non-decreasing (prove!). Let a_N be the first term in the sequence that lies within this interval. There can be no terms of the sequence beyond this interval since the sequence is non-decreasing (we would not be able to enumerate terms beyond this interval). Therefore

$$|a_i - a_j| < \epsilon \quad \text{for all } i, j \geq N.$$

This means the sequence is Cauchy, and so by $C3$, it converges. □