

Math 25a Solution Set #2 (Part A)

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October 05, 2001

Problem 1

Let W_1, W_2 be subspaces of a vector space V .

Part (a). Prove that $W_1 \cap W_2$ is also a subspace of V .

A subspace is a NONEMPTY subset of a vector space that is closed under addition and scalar multiplication. It is obvious that $W_1 \cap W_2 \subset W_1 \subset V$. To prove $W_1 \cap W_2 \neq \emptyset$ notice that $0 \in W_1$ and $0 \in W_2$, hence $0 \in W_1 \cap W_2$.

PLEASE MAKE SURE YOU UNDERSTAND WHY THIS IS A NECESSARY PART OF THE PROOF!

Now it is left to show that $W_1 \cap W_2$ is closed under addition and scalar multiplication to prove that it is a subspace of V . Consider any $x, y \in W_1 \cap W_2$ and $\alpha \in \mathbb{R}$. Because W_1 is a subspace and $x, y \in W_1$, we know that $x + y \in W_1$ and $\alpha \cdot x \in W_1$. Similarly, because W_2 is a subspace and $x, y \in W_2$, we know that $x + y \in W_2$ and $\alpha \cdot x \in W_2$. This tells us that $x + y \in W_1 \cap W_2$ and that $\alpha \cdot x \in W_1 \cap W_2$, which is precisely what we need to conclude $W_1 \cap W_2$ is closed under addition and scalar multiplication. ■

Part (b). We will prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.

⇐. If $W_1 \subset W_2$, then $W_1 \cup W_2 = W_2$ is a subspace of V . Similarly, if $W_2 \subset W_1$, then $W_1 \cup W_2 = W_1$ is a subspace of V .

⇒. We prove the contrapositive. Suppose W_1 and W_2 are not subsets of each other and $W_1 \cup W_2$ is a subspace of V . Pick any x, y such that $x \in W_1$, $y \in W_2$ and $x \notin W_2$, $y \notin W_1$. Now let $z = x + y$. Since $W_1 \cup W_2$ is a subspace, we know that $z \in W_1 \cup W_2$. If $z \in W_1$, then $y = z - x \in W_1$, and if $z \in W_2$, then $x = z - y \in W_2$. Both cases lead to contradiction with our choice of x, y , so we conclude our assumption is false, and either $W_1 \subset W_2$, or $W_2 \subset W_1$. ■

Problem 2

Let $T : V \rightarrow W$ be a linear map, and denote the image of T with $T(V)$, kernel of T with $\text{Ker}(T)$. **Part (a).** We will prove $T(V)$ is a subspace of W . First we notice $T(V) \neq \emptyset$, since $T(0) = 0$ and $0 \in V$ imply $0 \in T(V)$. Let $v_1, v_2 \in V$, hence $T(v_1), T(v_2) \in T(V)$. Since V is a vector space we know that $v_1 + v_2 \in V$ and therefore $T(v_1 + v_2) \in T(V)$. The fact that T is linear implies that $T(v_1) + T(v_2) = T(v_1 + v_2)$, so the sum of any two elements of $T(V)$ is itself an element of $T(V)$. Thus $T(V)$ is closed under addition. Now let α be any scalar. Since V is a vector space we know that $\alpha \cdot v_1 \in V$ and therefore $T(\alpha \cdot v_1) \in T(V)$. The fact that T is a linear map implies that $\alpha \cdot T(v_1) = T(\alpha \cdot v_1)$ which is itself an element of $T(V)$, so T is closed under scalar multiplication as well as addition and is a subspace. ■

Part (b). Here we prove $\text{Ker}(T)$ is a subspace of V .

First we notice $\text{Ker}(T) \neq \emptyset$, since $T(0) = 0$ implies $0 \in \text{Ker}(T)$. Let $v_1, v_2 \in \text{Ker}(T)$, hence $T(v_1) = T(v_2) = 0$. Because T is a linear map, we have $T(v_1) + T(v_2) = T(v_1 + v_2) = 0$. Thus $v_1 + v_2 \in \text{Ker}(T)$ and we see that $\text{Ker}(T)$ is closed under addition.

Now let α be any scalar. Since $T(v_1) = 0$, it follows that $\alpha \cdot T(v_1) = 0$. Because T is a linear map, we have $T(\alpha \cdot v_1) = \alpha \cdot T(v_1) = 0$. Thus $\alpha \cdot v_1 \in \text{Ker}(T)$, so $\text{Ker}(T)$ is closed under scalar multiplication as well as addition and is a subspace. ■

Problem 3

Part (a). Let $S, T \in \text{Lin}(V, W)$, $v \in V$, and $\alpha, \beta \in \mathbb{R}$. Define addition in $\text{Lin}(V, W)$ as $(S + T)(v) := S(v) + T(v)$. Since S and T are linear maps, we can show that their sum is also a linear map. For $v_1, v_2 \in V$, we know that $S(v_1) + S(v_2) = S(v_1 + v_2)$ and $T(v_1) + T(v_2) = T(v_1 + v_2)$. Therefore we have

$$\begin{aligned}(S + T)(v_1 + v_2) &= S(v_1 + v_2) + T(v_1 + v_2) \\ &= S(v_1) + S(v_2) + T(v_1) + T(v_2) \\ &= [S(v_1) + T(v_1)] + [S(v_2) + T(v_2)] \\ &= (S + T)(v_1) + (S + T)(v_2).\end{aligned}$$

Furthermore, $S(\alpha \cdot v) = \alpha \cdot S(v)$ and $T(\alpha \cdot v) = \alpha \cdot T(v)$, so

$$\begin{aligned}(S + T)(\alpha \cdot v) &= S(\alpha \cdot v) + T(\alpha \cdot v) \\ &= \alpha \cdot S(v) + \alpha \cdot T(v) \\ &= \alpha \cdot [S(v) + T(v)] \\ &= \alpha \cdot (S + T)(v)\end{aligned}$$

and we see that $(S + T)$ is a linear map and thus $\text{Lin}(V, W)$ is closed under

addition, so our addition in $Lin(V, W)$ is well-defined. Notice that the necessity of showing the linearity of $S + T$ as we defined it!

Define scalar multiplication in $Lin(V, W)$ as $(\alpha \cdot S)(v) := \alpha \cdot S(v)$. Using the same properties and relationships as above, we have

$$\begin{aligned} (\alpha \cdot S)(v_1 + v_2) &= \alpha \cdot S(v_1 + v_2) \\ &= \alpha \cdot [S(v_1) + S(v_2)] \\ &= \alpha \cdot S(v_1) + \alpha \cdot S(v_2) \\ &= (\alpha \cdot S)(v_1) + (\alpha \cdot S)(v_2) \end{aligned}$$

and

$$\begin{aligned} (\alpha \cdot S)(\beta \cdot v) &= \alpha \cdot S(\beta \cdot v) \\ &= \alpha \cdot [\beta \cdot S(v)] \\ &= \beta \cdot [\alpha \cdot S(v)] \\ &= \beta \cdot (\alpha \cdot S)(v). \end{aligned}$$

Therefore $(\alpha \cdot S)$ is a linear map and $Lin(V, W)$ is closed under scalar multiplication, hence our scalar multiplication is well defined in $Lin(V, W)$. It also satisfies the eight axioms of vector spaces, of which we will prove axioms 1 (existence of zero), 2 (existence of the inverse), and 4 (associativity).

First, we'll prove axiom 1 (existence of zero). Since W is a vector space, we know that there exists an additive identity $0_W \in W$. Note that $S(v)$ is also in W for any $v \in V$, hence $0_W + S(v) = S(v)$. Define $0 \in Lin(V, W)$ to be the linear map where $v \mapsto 0_W$ for all $v \in V$. We see that $0(v) + S(v) = 0_W + S(v) = S(v)$, so 0 is the additive identity in $Lin(V, W)$.

Next, we'll prove axiom 2 (existence of the inverse). We want to show that for any linear map $S \in Lin(V, W)$ there exists a map $-S \in Lin(V, W)$ such that $S + (-S) = 0$, or, for all $v \in V$, $S(v) + (-S)(v) = 0(v) = 0_W$. The proof is straightforward. Since $S(v) \in W$ and W is known to be a vector space, there is an inverse $-S(v) \in W$ such that $S(v) + [-S(v)] = 0_W$. It follows that the additive inverse $-S \in Lin(V, W)$ is the function mapping $v \mapsto -S(v)$ for all $v \in V$. In other words, $(-S)(v) = -S(v)$, where it can be easily verified (in the way above) that $-S$ is linear when S is.

Finally, we'll prove axiom 4 (associativity). Given any three linear maps $S, T, U \in Lin(V, W)$, we have (for all $v \in V$)

$$\begin{aligned} (S + T)(v) + U(v) &= S(v) + T(v) + U(v) \\ &= S(v) + (T + U)(v) \end{aligned}$$

■

Part (b). Any linear map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be expressed as a matrix $A \in Mat(m, n)$. The matrices in $(\mathbb{R}^n)^*$ map from \mathbb{R}^n to \mathbb{R} and are therefore $1 \times n$ matrices. Any such matrix is also the transpose of some vector in \mathbb{R}^n .

Problem 4

Let $(v_1, w_1), (v_2, w_2) \in V \times W$, $\alpha \in \mathbb{R}$. Define addition in $V \times W$ as

$$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2)$$

and scalar multiplication as

$$\alpha \cdot (v_1, w_1) := (\alpha \cdot v_1, \alpha \cdot w_1).$$

Since V and W are vector spaces, we know that $v_1 + v_2 \in V$ and $w_1 + w_2 \in W$, so $(v_1 + v_2, w_1 + w_2) \in V \times W$. Furthermore, $\alpha \cdot v_1 \in V$ and $\alpha \cdot w_1 \in W$, so $(\alpha \cdot v_1, \alpha \cdot w_1) \in V \times W$. Therefore $V \times W$ is closed under addition and scalar multiplication. It also satisfies all the axioms of vector spaces; we will prove axioms 1, 2, and 5 (multiplicative identity) below.

First, we'll prove axiom 1 (existence of zero). Since V and W are vector spaces, there exist vectors $0_V \in V$, $0_W \in W$ such that if $v \in V$ and $w \in W$ we have $0_V + v = v$ and $0_W + w = w$. Therefore the additive identity in $V \times W$ is $0 = (0_V, 0_W)$, since $(0_V, 0_W) + (v, w) = (0_V + v, 0_W + w) = (v, w)$.

Next, we'll prove axiom 2 (existence of the inverse). Since V and W are vector spaces, we know that for all $v \in V$ and $w \in W$ there exist vectors $-v \in V$, $-w \in W$ such that $v + (-v) = 0_V$ and $w + (-w) = 0_W$. Therefore for any $(v, w) \in V \times W$, the additive inverse is $-(v, w) = (-v, -w)$, since $(v, w) + (-v, -w) = (v + (-v), w + (-w)) = (0_V, 0_W) = 0$.

Finally, we will prove axiom 5 (multiplicative identity). Since V and W are vector spaces, we know that $1 \cdot v = v$ and $1 \cdot w = w$ for any $v \in V$, $w \in W$. This gives us $1 \cdot (v, w) = (1 \cdot v, 1 \cdot w) = (v, w)$. ■

Problem 5

Let $W \subset V$ be a subspace of a vector space V . We define a relation \sim in V by $v_1 \sim v_2$ if $v_1 - v_2 \in W$.

Part (a). To show that \sim is an equivalence relation, we will prove that it satisfies the properties of reflexivity, symmetry, and transitivity.

First we prove reflexivity. For any $v \in V$ we have $v - v = v + (-v) = 0 \in W$, therefore $v \sim v$.

Next we will prove symmetry. Let $v_1, v_2 \in V$ such that $v_1 \sim v_2$, hence $v_1 - v_2 \in W$. Since W is a vector space, we know it is closed under scalar multiplication, so $v_2 - v_1 = -(v_1 - v_2) \in W$, and we see that $v_2 \sim v_1$.

Finally, we will prove transitivity. Let $v_1, v_2, v_3 \in V$ such that $v_1 \sim v_2$ and $v_2 \sim v_3$, hence $v_1 - v_2, v_2 - v_3 \in W$. Since W is closed under addition, we get $(v_1 - v_2) + (v_2 - v_3) = v_1 + (-v_2) + v_2 + (-v_3) = v_1 + 0 + (-v_3) = v_1 - v_3 \in W$ so we see that $v_1 \sim v_3$. Thus \sim is an equivalence relation in V . ■

Part (b). Let $a_1, a_2, b_1, b_2 \in V$ such that $a_1 \sim a_2$ and $b_1 \sim b_2$, hence $(a_1 - a_2), (b_1 - b_2) \in W$. Also, let $\alpha \in \mathbb{R}$. We will define addition in V/W by

$[a_1] + [b_1] = [a_1 + b_1]$ and scalar multiplication by $\alpha \cdot [a_1] = [\alpha \cdot a_1]$. Since V is a vector space we know that $(a_1 + b_1), (\alpha \cdot a_1) \in V$, so $[a_1 + b_1], [\alpha \cdot a_1] \in V/W$ and the set V/W is closed under multiplication and addition. However, we must still check that these operations are well-defined.

To prove that addition is well-defined, we will demonstrate that $[a_1 + b_1] = [a_2 + b_1] = [a_1 + b_2] = [a_2 + b_2]$. It is sufficient to prove that $a_1 + b_1 \sim a_2 + b_1 \sim a_1 + b_2 \sim a_2 + b_2$ because we know from problem set 1, part A, problem 6 (1.A.6) that if two equivalence classes contain any overlapping elements then they must be equal. We know that $(a_1 + b_1) - (a_2 + b_1) = (a_1 - a_2) \in W$, thus $(a_1 + b_1) \sim (a_2 + b_1)$. We also know that $(a_1 + b_1) - (a_1 + b_2) = (b_1 - b_2) \in W$, thus $(a_1 + b_1) \sim (a_1 + b_2)$. Finally, we know that $(a_1 + b_2) - (a_2 + b_2) = (a_1 - a_2) \in W$, thus $(a_1 + b_2) \sim (a_2 + b_2)$. Since \sim is transitive, we see that all $a_i + b_i$ are equivalent and that addition is well-defined.

To prove that scalar multiplication is well-defined, we will show that $[\alpha \cdot a_1] = [\alpha \cdot a_2]$. We see that $\alpha \cdot a_1 - \alpha \cdot a_2 = \alpha \cdot (a_1 - a_2) \in W$ since $(a_1 - a_2) \in W$ and W is closed under scalar multiplication. Therefore $\alpha \cdot a_1 \sim \alpha \cdot a_2$ and we see from the result in 1.A.6 that $[\alpha \cdot a_1] = [\alpha \cdot a_2]$. Thus scalar multiplication is well-defined. The set V/W is a vector space because it also satisfies all eight vector space axioms, of which we will prove axioms 1, 2, and 5.

First, we'll prove axiom 1 (existence of zero), where we use 0 to denote the additive identity of V . For any $v \in V$, we see that $[v] + [0] = [v + 0] = [v]$. Therefore $[0]$ is the additive identity in V/W .

Next, we'll prove axiom 2 (existence of the inverse). Since V is a vector space, we know that for all $v \in V$ there exists an inverse $-v \in V$ such that $v + (-v) = 0$. We have $[v] + [-v] = [v + (-v)] = [0]$, so $[-v]$ is the additive inverse of $[v]$ in V/W .

Finally, we will prove axiom 5 (multiplicative identity). Since V is a vector space, we know that $1 \cdot v = v$. Then we have $1 \cdot [v] = [1 \cdot v] = [v]$. ■

Problem 6

Part (a). We show that the matrix transpose satisfies $(A \cdot B)^t = B^t \cdot A^t$. Using the definition of matrix multiplication, the proof is straightforward:

$$\begin{aligned} (A \cdot B)_{ij} &= \sum_k A_{ik} B_{kj} \\ (A \cdot B)^t_{ji} &= \sum_k A_{ik} B_{kj} \\ (A \cdot B)^t_{ij} &= \sum_k A_{jk} B_{ki} \\ &= \sum_k B^t_{ik} A^t_{kj} \\ &= (B^t \cdot A^t)_{ij} \end{aligned}$$

It is crucial to use consistent notation in proofs like this one. Here we have used M_{ij} to denote the i, j entry of M . ■

Part (b). Here we show that if a matrix A is invertible, then A^t is invertible as well, and $(A^t)^{-1} = (A^{-1})^t$.

First we notice that an invertible matrix must be a square matrix. To see why this is so, assume that $A \in \text{Mat}(n, m)$, where $n \neq m$, is invertible. Then $A^{-1} \in \text{Mat}(m, n)$ and we have $I_m = A^{-1}A = AA^{-1} = I_n$, but since I_n and I_m are different sizes they cannot be equal, so A cannot be invertible unless it is a square matrix.

Now, we can prove that $(A^t)^{-1} = (A^{-1})^t$ by using the result from part (a). (Note that the identity matrix I is a diagonal matrix, so $I = I^t$.) By transposing each term in $A \cdot A^{-1} = A^{-1} \cdot A = I_n$ according to part (a) we get that $(AA^{-1})^t = (A^{-1}A)^t = I_n^t$ implies $(A^{-1})^t(A)^t = A^t(A^{-1})^t = I_n$, so we conclude A^t is invertible with inverse $(A^{-1})^t$. ■