Math 25a Problem Set #2b Solutions

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Problem 1

Part (a). To prove that $(\mathbb{R}^n, d_2)$ is a metric space, we must show that d_2

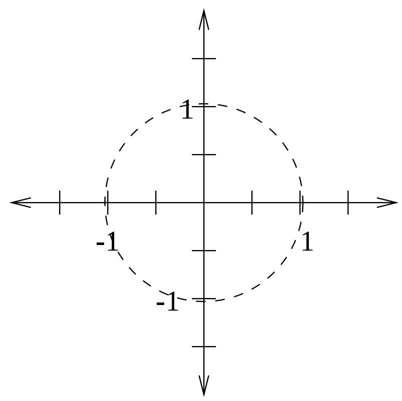

Figure 1: The unit ball under the d_2 norm.

satisfies the four properties of a metric over $\mathbb{R}^n$. For this problem, let $p, q, r \in \mathbb{R}^n$. First we will prove that $d_2(p,q) \geq 0$. We have $d_2(p,q) = \sqrt{\sum_{i=1}^n (p_i - q_i)^2}$, which is the principal square root of the sum of i non-negative numbers. This value is real and also non-negative, so we have $d_2(p,q) \geq 0$.

Next, we will show that $d_2(p,q) = 0 \iff p = q$. When $p = q$, we know that each $p_i = q_i$ and $d_2(p,q) = \sqrt{\sum_{i=1}^n (q_i - q_i)^2} = 0$. If, however, $p \neq q$, there is some $(p_i - q_i) \neq 0$ and therefore $d_2(p,q) \neq 0$. Thus the distance between p and q can be zero only if $p = q$.

We will now show that $d_2(p,q) = d_2(q,p)$. This is true because $(p_i - q_i)^2 = (-1)^2(q_i - p_i)^2 = (q_i - p_i)^2$. Therefore the sum in the expression of $d_2(p,q)$ equals that for $d_2(q,p)$.

Finally, we will show that $d_2(p,q) \leq d_2(p,r) + d_2(r,q)$. Note that the Euclidean norm of the vector $p - q$, denoted here as $|p - q|$, is given by

$$\sqrt{(p-q)\cdot(p-q)} = \sqrt{(p_1 - q_1)^2 + \cdots + (p_n - q_n)^2} = d_2(p,q).$$

We will use this to prove that $|p-q|^2 \leq (|p-r|+|r-q|)^2$, which will imply that $d_2(p,q) \leq d_2(p,r)+d_2(r,q)$. The derivation also makes use of the Schwarz inequality between the third and fourth steps below.

$$\begin{aligned}
|p-q|^2 &= (p-q)\cdot(p-q) \\
|p-q|^2 &= [(p-r)+(r-q)]\cdot[(p-r)+(r-q)] \\
|p-q|^2 &= (p-r)\cdot(p-r)+2(p-r)\cdot(r-q)+(r-q)\cdot(r-q) \\
|p-q|^2 &\leq |p-r|^2+2|p-r|\cdot|r-q|+|r-q|^2 \\
|p-q|^2 &\leq (|p-r|+|r-q|)^2
\end{aligned}$$

∎

Part (b). To prove that $(\mathbb{R}^n, d_1)$ is a metric space, we must show that d_1

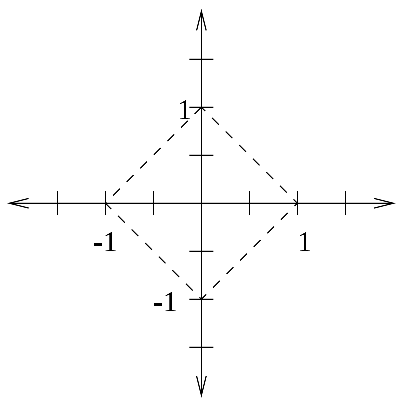

Figure 2: The unit ball under the d_1 norm.

satisfies the four properties of a metric over $\mathbb{R}^n$. For this problem, let $p,q,r \in \mathbb{R}^n$. First we will prove that $d_1(p,q) \geq 0$. We have $d_1(p,q) = \sum_{i=1}^n |p_i - q_i|$, which is the sum of i non-negative numbers. Consequently $d_1(p,q) \geq 0$.

Next, we will show that $d_1(p,q) = 0 \iff p = q$. When $p = q$, we know that each $p_i = q_i$ and $d_1(p,q) = \sum_{i=1}^n |q_i - q_i| = 0$. If, however, $p \neq q$, there is some $|p_i - q_i| > 0$ and therefore $d_1(p,q) \neq 0$ (since there can be no negative terms in the sum to counteract this positive term). Thus the distance between p and q can be zero only if $p = q$.

We will now show that $d_1(p,q) = d_1(q,p)$. This is true because $|p_i - q_i| = |-1|\cdot|q_i - p_i| = |q_i - p_i|$. Therefore the sum in the expression of $d_1(p,q)$ equals that for $d_1(q,p)$.

Finally, we will show that $d_1(p,q) \leq d_1(p,r) + d_1(r,q)$, using the triangle inequality for absolute values of sums in $\mathbb{R}$. For each $i \in \{1,2,\ldots,n\}$, we have

$$\begin{aligned}
|p_i - q_i| &= |p_i - r_i + r_i - q_i| \\
&\leq |p_i - r_i| + |r_i - q_i|.
\end{aligned}$$

Since this relationship is true for all i, it also holds for the sum over i:

$$\begin{aligned}\sum_{i=1}^{n}|p_i - q_i| &\leq \sum_{i=1}^{n}|p_i - r_i| + \sum_{i=1}^{n}|r_i - q_i| \\ d_1(p,q) &\leq d_1(p,r) + d_1(r,q)\end{aligned}$$

∎

Part (c). To prove that $(\mathbb{R}^n, d_\infty)$ is a metric space, we must show that

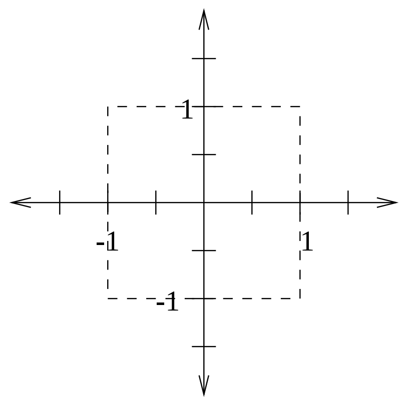

Figure 3: The unit ball under the d_∞ norm.

d_∞ satisfies the four properties of a metric over $\mathbb{R}^n$. For this problem, let $p, q, r \in \mathbb{R}^n$. First we will prove that $d_\infty(p,q) \geq 0$. We have $d_\infty(p,q) = |p_i - q_i|$ for some i, which is non-negative by the definition of the absolute value function in $\mathbb{R}$. Thus $d_\infty(p,q) \geq 0$.

Next, we will show that $d_\infty(p,q) = 0 \iff p = q$. When $p = q$, we know that each $p_i = q_i$ and

$$\begin{aligned}d_\infty(p,q) &= \max\{|p_i - q_i|\} \\ &= \max\{|q_i - q_i|\} \\ &= \max\{0, 0, \ldots, 0\} \\ &= 0.\end{aligned}$$

If, however, $p \neq q$, there is some $|p_i - q_i| > 0$ and therefore $d_\infty(p,q) = \max\{|p_i - q_i|\} \neq 0$. Thus the distance between p and q can be zero only if $p = q$.

We will now show that $d_\infty(p,q) = d_\infty(q,p)$. This is true because $|p_i - q_i| = |-1| \cdot |q_i - p_i| = |q_i - p_i|$, which means that $\max\{|p_i - q_i|\} = \max\{|q_i - p_i|\}$. Therefore $d_\infty(p,q) = d_\infty(q,p)$.

Finally, we will show that $d_\infty(p,q) \leq d_\infty(p,r) + d_\infty(r,q)$. Let $m \in \{1, \ldots, n\}$ such that the maximum value of $|p_i - q_i|$, which is $d_\infty(p,q)$, occurs at $i = m$.

Then we have

$$|p_m - q_m| \leq |p_m - r_m| + |r_m - q_m|.$$

Since $|p_m - r_m| \in \{|p_i - r_i|\}$, it is clear that $d_\infty(p, r) = \max\{|p_i - r_i|\} \geq |p_m - r_m|$. In like manner, we can show that $d_\infty(r, q) \geq |r_m - q_m|$. Thus we have

$$d_\infty(p, q) \leq d_\infty(p, r) + d_\infty(r, q)$$

as we hoped to prove. ∎

Part (d). To prove that $(C([0,1]), d_1)$ is a metric space, we must show that d_1 satisfies the four properties of a metric over $C([0,1])$. For this problem, let $p, q, r \in C([0,1])$. First we will prove that $d_1(p, q) \geq 0$. We have $d_1(p, q) = \int_0^1 |p(x) - q(x)|dx$, the integral of a non-negative function, so $d_1(p, q) \geq 0$ (by the expansoion of the integral as a Riemann sum.)

Next, we will show that $d_1(p, q) = 0 \iff p = q$. When $p = q$, we know that $p(x) - q(x) = 0$ on $[0,1]$ and we have $d_1(p, q) = \int_0^1 0 \cdot dx = 0$. If, however, $p \neq q$, then $|p(x_0) - q(x_0)| > 0$ for some $x_0 \in [0,1]$. Since p and q are continuous, their difference $p - q$ and its absolute value are also continuous. Thus if we let $\varepsilon = |p(x_0) - q(x_0)|$, the value of our function of integration at x_0, there exists a real number δ such that the function of integration lies within ε of $|p(x_0) - q(x_0)|$, and is therefore greater than zero, whenever $x \in (x_0 - \delta, x_0 + \delta)$. Thus we have

$$\int_{x_0 - \delta}^{x_0 + \delta} |p(x) - q(x)|dx > 0$$

and we know that the integral is non-negative on $[0, x_0 - \delta]$ and $[x_0 + \delta, 1]$, so we find that $d_1(p, q) > 0$. Thus $d_1(p, q)$ can be zero only if $p = q$.

We will now show that $d_1(p, q) = d_1(q, p)$. This is true because $|p(x) - q(x)| = |-1| \cdot |q(x) - p(x)| = |q(x) - p(x)|$, which means that $\int_0^1 |p(x) - q(x)|dx = \int_0^1 |q(x) - p(x)|dx$. Therefore $d_1(p, q) = d_1(q, p)$.

Finally, we will show that $d_1(p, q) \leq d_1(p, r) + d_1(r, q)$. We have

$$\begin{aligned}
d_1(p, q) &= \int_0^1 |p(x) - q(x)|dx \\
d_1(p, q) &= \int_0^1 |p(x) - r(x) + r(x) - q(x)|dx \\
d_1(p, q) &\leq \int_0^1 (|p(x) - r(x)| + |r(x) - q(x)|)\, dx \\
d_1(p, q) &\leq \int_0^1 |p(x) - r(x)|dx + \int_0^1 |r(x) - q(x)|dx \\
d_1(p, q) &\leq d_1(p, r) + d_1(r, q)
\end{aligned}$$

∎

Part (e). To prove that $(C([0,1]), d_\infty)$ is a metric space, we must show that d_∞ satisfies the four properties of a metric over $C([0,1])$. For this problem, let $p, q, r \in C([0,1])$. First we will prove that $d_\infty(p,q) \geq 0$. We have $d_\infty(p,q) = \max_x |p(x) - q(x)|$, which is non-negative by the definition of the absolute value function in $\mathbb{R}$. Thus $d_\infty(p,q) \geq 0$.

Next, we will show that $d_\infty(p,q) = 0 \iff p = q$. When $p = q$, we know that each $p(x) = q(x)$ and

$$\begin{aligned} d_\infty(p,q) &= \max_x |p(x) - q(x)| \\ &= \max_x |q(x) - q(x)| \\ &= 0. \end{aligned}$$

If, however, $p \neq q$, there is some $|p(x) - q(x)| > 0$ and therefore $d_\infty(p,q) = \max_x |p(x) - q(x)| \neq 0$. Thus $d_\infty(p,q)$ can be zero only if $p = q$.

We will now show that $d_\infty(p,q) = d_\infty(q,p)$. We see that this is true because $|p(x) - q(x)| = |-1| \cdot |q(x) - p(x)| = |q(x) - p(x)|$, hence $\max_x |p(x) - q(x)| = \max_x |q(x) - p(x)|$. Therefore $d_\infty(p,q) = d_\infty(q,p)$.

Finally, we will show that $d_\infty(p,q) \leq d_\infty(p,r) + d_\infty(r,q)$. Let $m \in [0,1]$ such that the maximum value of $|p(x) - q(x)|$, which is $d_\infty(p,q)$, occurs at $x = m$. Then we have

$$|p(m) - q(m)| \leq |p(m) - r(m)| + |r(m) - q(m)|.$$

Since $|p(x) - r(x)| = |p(m) - r(m)|$ for some $x \in [0,1]$, it is clear that $d_\infty(p,r) = \max_x |p(x) - r(x)| \geq |p(m) - r(m)|$. In like manner, we can show that $d_\infty(r,q) \geq |r(m) - q(m)|$. Thus we have

$$d_\infty(p,q) \leq d_\infty(p,r) + d_\infty(r,q)$$

as we hoped to prove. ∎

Problem 2

Let $p, q, r \in T$, hence also $p, q, r \in S$. Define the metric d_T by $d_T(p,q) = d(p,q)$. To prove that (T, d_T) is a metric space, we must show that d_T satisfies the four properties of a metric over T. Note that we must also assume that T is a nonempty subset of S.

It is straightforward to show that d_T satisfies the four properties of a metric over T. Because (S,d) is a metric space, we know that $d(p,q) \geq 0$, therefore $d_T(p,q) = d(p,q) > 0$ as well. We know that $d(p,q) = 0 \iff p = q$. Since $d(p,q) = d_T(p,q)$, this means that $d_T(p,q) = 0 \iff p = q$. We know that $d(p,q) = d(q,p)$, therefore $d_T(p,q) = d(p,q) = d(q,p) = d_T(q,p)$. Finally, we know that $d(p,q) \leq d(p,r) + d(r,q)$, so by substitution we see that $d_T(p,q) \leq d_T(p,r) + d_T(r,q)$. Thus (T, d_T) is a metric space for all nonempty subsets $T \subset S$. ∎

Problem 3

See figures 1,2, and 3 in problem 1.

Problem 4

Part (a). Take any sequence $\{a_i\}$ that converges to a with respect to d_1, hence for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $a_i \in B^1_\varepsilon(a)$ whenever $i > N$. Since we are given the fact that d_1 and d_2 are equivalent, we know that there exists an open ball $B^2_R(a) \subset B^1_\varepsilon(a)$ and, furthermore, there exists another open ball $B^2_{\varepsilon_0}(a) \subset B^2_R$. Now for any $\varepsilon_0 > 0$ there exists an $M \in \mathbb{N}$ such that $a_i \in B^1_{\varepsilon_0}(a)$ whenever $i > M$. Note that because $a_i \in B^2_{\varepsilon_0}(a) \subset B^2_R(a)$, we have $a_i \in B^2_R(a)$ for any $i > M$ and we see that the sequence $\{a_i\}$ converges with respect to d_2, as we hoped to prove.

Similarly, we can exchange all the 1s and 2s in the previous paragraph to prove that a sequence that converges with respect to d_2 also converges with respect to d_1. ∎

Part (b). We will first prove $d_1 \sim d_2$ by showing that $\forall x \in \mathbb{R}$ and $\forall \epsilon > 0$, there exist δ_1, δ_2 such that

$$B^{d_1}_{\delta_1}(x) \supset B^{d_2}_{\epsilon}(x) \supset B^{d_1}_{\delta_2}(x)$$

Specifically, we make $\delta_1 := \sqrt{2}\epsilon$ and $\delta_2 := \epsilon$.

We next prove that $d_\infty \sim d_2$ by showing that $\forall x \in \mathbb{R}$ and $\forall \epsilon > 0$, there exist δ_1, δ_2 such that

$$B^{d_\infty}_{\delta_1}(x) \supset B^{d_2}_{\epsilon}(x) \supset B^{d_\infty}_{\delta_2}(x)$$

Specifically, we make $\delta_1 := \epsilon$ and $\delta_2 := \frac{\sqrt{2}}{2}\epsilon$.

Part (c). For $i \in \{1, 2, \dots\}$, define the function $f_i \in C([0,1])$ so that its graph consists of a line segment from the point $A = (0,0)$ to the point $B = \left(\left[\frac{1}{2}\right]^i, 2^i\right)$, a segment B to the point $C = \left(\left[\frac{1}{2}\right]^{i-1}, 0\right)$, and a segment from C to the point $D = (1,0)$. The formal definition of our function is

$$f_i = \begin{cases} 4^i \cdot x & \text{if } 0 \le x < \left(\frac{1}{2}\right)^i \\ -4^i \cdot x + 2^{i+1} & \text{if } \left(\frac{1}{2}\right)^i \le x < \left(\frac{1}{2}\right)^{i-1} \\ 0 & \text{if } \left(\frac{1}{2}\right)^{i-1} \le x \le 1. \end{cases}$$

For a sequence to converge in $C([0,1])$ with respect to d_1, the sequence of integrals from $[0,1]$ of successive functions must converge. In our example, this integral for the ith term of the sequence is the area of triangle ABC above, which is $\frac{1}{2} \cdot \left(\frac{1}{2}\right)^{i-1} \cdot 2^i = \left(\frac{1}{2}\right)^i \cdot 2^i = 1$. Thus the sequence of integrals is $\{1, 1, 1, \dots\}$, which converges to 1. Therefore $\{f_i\}$ converges with respect to d_1.

For a sequence to converge in $C([0,1])$ with respect to d_∞, the sequence of maxima of successive functions must converge. We see that the maximum value of f_i above is always the height 2^i of triangle ABC. However, the sequence $\{2^i\}$ increases without bound as $i \to \infty$, so $\{f_i\}$ does not converge with respect to d_∞. Since this sequence in $C([0,1])$ converges with respect to d_1 but not with respect to d_∞, we know that these two metrics are not equivalent. ∎

Problem 5

We want to prove that $(\mathbb{R}^n, d_2)$ is complete, but since d_2 and d_1 are equivalent metrics on $\mathbb{R}^n$ we know that all sequences that converge with respect to d_1 do so with respect to d_2 and it is therefore sufficient to prove that $(\mathbb{R}^n, d_1)$ is complete.

We know that a Cauchy sequence in $\mathbb{R}$ converges, so if we look at a single coordinate of any two vectors in any Cauchy sequence $\{a_i\}$ in $\mathbb{R}^n$ we have that for all $\varepsilon > 0$ there exists N_k such that $|(a_i)_k - (a)_k| < \varepsilon/n$ whenever $i > N_k$; note that a_k is the limit of the Cauchy sequence $\{(a_i)_k\}$ in $\mathbb{R}$. Therefore the sum of the absolute differences of all these coordinates is

$$\begin{aligned} \sum_{k=1}^{n} |(a_i)_k - (a)_k| &< \sum_{k=1}^{n} \frac{\varepsilon}{n} \\ |(a_i)_1 - (a)_1| + \cdots + |(a_i)_n - (a)_n| &< n \cdot \frac{\varepsilon}{n} \\ d_1(a_i, a) &< \varepsilon \end{aligned}$$

whenever $i > N = \max\{N_1, \ldots, N_n\}$. This is the definition of $\{a_i\}$ converging to a in $(\mathbb{R}^n, d_1)$, so because d_1 and d_2 are equivalent metrics in $\mathbb{R}^n$ we can say that any Cauchy sequence $\{a_i\}$ in $(\mathbb{R}^n, d_2)$ also converges. ∎

Problem 6

Part (a). Assume that the contraction map has multiple fixed points $p, q \in S$, hence $f(p) = p$ and $f(q) = q$. Then we have

$$\begin{aligned} d(f(p), f(q)) &< Cd(p,q) \\ d(p,q) &< Cd(p,q) < d(p,q) \end{aligned}$$

which is a contradiction. ∎

Part (b). We begin by defining $d_i := d(a_i, a_{i+1})$ and noting that $\forall j > N$:

$$\begin{aligned} d(a_N, a_j) &= d(a_N, a_{N+1}) + d(a_{N+1}, a_{N+2}) + \cdots + d(a_{j-1}, a_j) \\ &= d_N + d_{N+1} + \cdots + d_{j-1} \\ &\leq d_0 \sum_{i=N}^{j} C^i \\ &\leq d_0 (j - N) C^j \end{aligned}$$

Because $C < 1$,

$$\leq d_0 (j - N) C^N$$

We will thus, attempt to prove that $\forall \epsilon > 0$ there exists N such that $d_0(j - N)C^j < \epsilon/2$ for any $j > N$. (We note that if this implies the Cauchy condition, as $d(a_i, a_j) \leq d(a_i, a_N) + d(a_j, a_N) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.)

If $d_0 = 0$ or $C = 0$, then the sequence is constant (at least after the first term) and therefore trivially Cauchy. Otherwise, we select $N > log_C(\frac{\epsilon}{2d_0})$ which is well-defined by our conditions on C and d_0, and clearly guarantees that $d_0(j - N)C^j < \epsilon/2$ for any $j > N$. ∎

Part (c). We know from part (b) that $\{a_i\}$ converges to some value $a \in S$. We want to prove that $f(a) = a$ and hence a is a fixed point of f. We will show that $\{a_i\}$ also converges to $f(a)$, allowing us to conclude that $f(a) = a$. So let $\epsilon > 0$. By the convergence of $\{a_i\}$, we know that $\exists N_1$ such that $d(a_i, a) \leq \frac{\epsilon}{2}$ for all $i > N$ and by the Cauchyness of $\{a_i\}$, we know that $\exists N_2$ such that $d(a_i, a_j) \leq \frac{\epsilon}{2C}$ for all $i, j > N$. Select $j > Max(N_1, N_2) + 2$. Then

$$\begin{aligned} d(a_j, f(a)) &\leq d(a_j, f(a_j)) + d(f(a_j), f(a) \\ &\leq Cd(a_{j-1}, a_j) + d(a_{j+1}, f(a)) \\ &\leq C\frac{\epsilon}{2C} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as required. ∎