# MATH 25A – PROBLEM SET #3
# DUE FRIDAY OCTOBER 5

## 1. Part A

(1) Problem 0.5.5 in the textbook.
(2) Problem 0.4.6 in the textbook.
(3) Prove that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$ if and only if for any sequence $\{a_i\}$ in $\mathbb{R}^n$ such that $\lim_{i\to\infty} a_i = a$, we have that $\lim_{i\to\infty} f(a_i) = f(a)$.
(4) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ is continuous at $f(a)$ then $g \circ f$ is continuous at $a$.
(5) Let $f : \mathbb{R} \to \mathbb{R}$ be

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \end{cases}$$

Here $x = \frac{p}{q}$ means that $p$ and $q > 0$ are integers with no common divisors. Prove that $f$ is continuous at all irrational points and $f$ is not continuous at any rational point.
(6) A function $f : [a, b] \to \mathbb{R}$ is said to be *convex* if for any $u, v \in [a, b]$ and any $0 \leq \lambda \leq 1$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v).$$

Prove that a convex function is continuous in the open interval $(a, b)$. Does it have to be continuous at the endpoints? (Hint: To prove continuity at a point $c \in (a, b)$, draw two lines in $\mathbb{R}^2$: one through the points $(a, f(a))$ and $(c, f(c))$, and another line through $(b, f(b))$ and $(c, f(c))$. These lines divide the plane into four quadrants. The graph of $f$ has to lie in two of the four quadrants.)

## 2. Part B

**Definition.** A *normed space* $(V, \|\cdot\|)$ consists of a vector space $V$ together with a function $\|\cdot\| : V \to \mathbb{R}$ that assigns to a vector $v \in V$ its *norm* $\|v\|$, satisfying the following properties for all $v, w \in V$, $\alpha \in \mathbb{R}$:

- $\|v\| \geq 0$.
- $\|v\| = 0$ if and only if $v = 0$.
- $\|\alpha v\| = |\alpha|\|v\|$.
- $\|v + w\| \leq \|v\| + \|w\|$.

(1) Prove that if $(V, \|\cdot\|)$ is a normed space, then

$$d(v, w) = \|v - w\|$$

is a metric on $V$.
(2) Consider the five metrics defined in Homework #2. Show that each metric comes from a norm. Prove that these really are norms.
(3) Define the norm of a matrix $A \in Mat(n, m)$ by

$$\|A\| = \sup_{0 \neq v \in \mathbb{R}^m} \frac{|Av|}{|v|}.$$

Prove that $\|A\|$ exists for any $A$ and that this defines a norm on $Mat(n, m)$.

(4) A subset $S \subset V$ of a vector space is called *convex* if for any $u, v \in S$ and for any $0 \le \lambda \le 1$, we have that $\lambda u + (1 - \lambda)v \in S$. If $\| \cdot \|$ is a norm on $V$, show that the open ball with radius $R > 0$

$$B_R = \{v \in V | \|v\| < R\}$$

is convex.

(5) Let $\| \cdot \|_\alpha$ and $\| \cdot \|_\beta$ be two norms on a vector space $V$. We say that $\| \cdot \|_\alpha \le \| \cdot \|_\beta$ is there exists a constant $C > 0$ such that

$$\|v\|_\alpha \le C\|v\|_\beta$$

for any $v \in V$. The two norms are said to be equivalent if $\| \cdot \|_\alpha \le \| \cdot \|_\beta$ and $\| \cdot \|_\beta \le \| \cdot \|_\alpha$.

(a) Prove that $\| \cdot \|_\alpha \le \| \cdot \|_\beta$ if and only if for some $C > 0$

$$B_1^\beta \subset B_C^\alpha.$$

Here $B_R^i$ is the open ball of radius $R$ with respect to the norm $\| \cdot \|_i$, $i = \alpha, \beta$.

(b) Prove that two norms are equivalent if and only if they define equivalent metrics.

(c) Prove that the three norms on $\mathbb{R}^n$ defined in problem (2) are all equivalent. (You may quote results from previous homeworks.)

(6) The goal of this exercise is to prove that all norms on $\mathbb{R}^n$ are equivalent by showing that an arbitrary norm $\| \cdot \|_\alpha$ on $\mathbb{R}^n$ is equivalent to the norm $\| \cdot \|_1$ defined in problem (2) (the "sum of absolute values" norm).

(a) Let $M = \max_i \|\vec{e}_i\|_\alpha$. Using triangle inequality prove that for any $v \in \mathbb{R}^n$

$$\|v\|_\alpha \le M\|v\|_1.$$

(b) Prove that the function $\| \cdot \|_\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous.

(c) Let $S = \{v \in \mathbb{R}^n | \|v\|_1 = 1\}$ be the unit sphere with respect to the norm $\| \cdot \|_1$ (so it is actually the boundary of a "diamond"). We consider the norm $\| \cdot \|_\alpha$ restricted to $S$:

$$\| \cdot \|_\alpha : S \to \mathbb{R}.$$

We will prove in class that such a function (a continuous function on a compact set) achieves its minimum $m$ at some point $u \in S$. Assuming this, prove that $m \ne 0$ and

$$\|v\|_1 \le \frac{1}{m}\|v\|_\alpha$$

for any $v \in \mathbb{R}^n$.