Problem Set 3, Part A Solution Set

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1. Question 1

Solution. We know the function f has range $[a, b]$. So $f(a) \geq a$ and $f(b) \leq b$. If $f(a) = a$ or $f(b) = b$, we are done. Otherwise, define the function $g(x) = f(x) - x$. Since f is continuous, and so is the identity, $g(x)$ is continuous (it is the difference of two continuous functions). From what we said before, $g(a) > 0$ and $g(b) < 0$. By the intermediate value theorem, there exists $c \in [a, b]$ such that $g(c) = 0$. For this c, $f(c) = c$ by construction. $\square$

2. Question 2

Remark. This problem has hard. I had to do it two years ago and I remember the pain. Overall, those who attempted it did quite well. I was looking for fairly rigorous proofs for parts (a) and (b). The other two parts did not have to be as water-tight to get you full credit.

Solution. (a) *(due to Michael Schnall-Levin)* Let A, B and $\sqrt{A^2 + B^2}$ be the sides of the original triangle and α, β and $\pi/2$ be their corresponding angles. The first iteration $x_0 \to x_1$ splits the triangle into 2 triangles similar to the original. One of the triangles is in ratio $\cos\alpha$ with respect to the original triangle, and the other is in ratio $\cos\beta$. At the next iteration, $x_1 \to x_2$, whether we take a right or a left, we again create 2 triangles similar to the one created in our first iteration. This continues for every iteration. For the i^{th} iteration, the length of the altitude $d(x_i, x_{i+1})$ is either $(\cos\alpha)d(x_{i-1}, x_i)$ or $(\cos\beta)d(x_{i-1}, x_i)$, depending on whether we took a right or left.

Now, Since we are dealing with right triangles, $0 < \cos\alpha < 1$, and $0 < \cos\beta < 1$. Let $r = \max\{\cos\alpha, \cos\beta\}$. It follows that $d(x_i, x_{i+1}) \leq r^i d(x_0, x_1)$, where $d(x_0, x_1) := D$ is the altitude of the original triangle.

By use of the triangle inequality we have, for $j > i$

$$d(x_i, x_j) \leq d(x_i, x_{i+1}) + \cdots + d(x_{j-1}, x_j)$$
$$d(x_i, x_j) \leq r^i D(1 + \cdots + r^{j-i-1}).$$

But $(1 + \cdots + r^{j-i-1}) < 1/(1-r)$ because the sum is a finite geometric progression and $r < 1$. Hence

$$d(x_i, x_j) \leq \frac{r^i D}{1-r} \quad \text{for } j > i.$$

Thus the sequence $\{x_i\}$ of points in $\mathbb{R}^2$ is Cauchy. This follows because given any $\epsilon > 0$ we can find an N such that $r^i < \epsilon(1-r)/D$ whenever $i > N$ which means $d(x_i, x_j) < \epsilon$ for $i, j > N$. Since $\mathbb{R}^2$ is complete under this metric (as we proved in the last problem set), the sequence $\{x_i\}$ converges.

(b) We'll follow the hint. Let $p = .1000\ldots$ and $q = .0111\ldots$. We'll show

$$\lim_{n\to\infty} x_n(p) = \lim_{n\to\infty} x_n(q).$$

Take a look at the file with hand-drawn figures. Like last time, let $d(x_0(p), x_1(p)) := D$. From trigonometry, we can see that $d(x_2(p), x_0(p)) = D\sin\alpha$, and $d(x_3(p), x_0(p)) = D\sin^2\alpha$, $\ldots$, and (by induction) $d(x_i(p), x_0(p)) = D\sin^{i-1}\alpha$. Since we have a right-triangle, $0 < \sin\alpha < 1$. Then for a given $\epsilon > 0$, we can choose large enough i so that $d(x_i(p), x_0(p)) < \epsilon$, which is to say, the sequence $\{x_i(p)\}$ converges to $x_0(p)$.

By a very similar argument, we find that $d(x_i(q), x_0(q)) < D\cos^{i-1}\alpha$ (see the figure again), and $0 < \cos\alpha < 1$. So $\{x_i(q)\}$ converges to $x_0(q)$. But $x_0(p) = x_0(q)$, so the sequences both converge to the same limit.

Now consider a general number $t \in [0,1]$ written in two different ways, t_1 and t_2. These two numbers will follow the same path until their k^{th} digits disagree. At this point, the situation we have just discussed will take over, with $x_0(p) = x_k(t_1)$ and $x_0(q) = x_k(t_2)$. We conclude that both numerical representations of t will converge to the same point.

(c) To prove continuity at a point t, we want to show that given $\epsilon > 0$, there is a δ such that

$$|\gamma(s) - \gamma(t)| < \epsilon \quad \text{whenever } |s - t| < \delta,$$

where the absolute value signs denote the normal distance in $\mathbb{R}^2$. Suppose s and t agree of the first k digits of their binary expansions. Consider the triangle that you get once you reach $x_k(s) = x_k(t)$. Denote this triangle's hypothenuse by h_k. By construction, $\lim_{n\to\infty} x_n(s) = \gamma(s)$ and $\lim_{n\to\infty} x_n(s) = \gamma(t)$ lie inside or on this triangle. The furthest appart these points could be is h_k. Hence, choose δ small enough so that the first k digits of the binary expansion agree *and* $h_k < \epsilon$. Then $|\gamma(s) - \gamma(t)| < h_k < \epsilon$ for this choice of δ and so γ is continuous.

(d) *(due to Emily Kendall)* Let a be a point in our triangle. We'll construct t such that $\gamma(t) = a$. First, draw the altitude x_0x_1. If a lies to the left of x_0x_1, make the first digit of t a 1. If a lies to the right of x_0x_1, make the first digit of t a 0. If a lies on x_0x_1, we can make the first digit either a 1 or a 0.

Continue in this manner, choosing digits so that a stays always inside or on the triangle whose altitude we are drawing. The areas of successive triangles continue to decrease by either $\cos^2\alpha$ or $\cos^2\beta$. Thus we'll get convergence and since every triangle constains a, we have a t such that $\gamma(t) = a$, thus proving surjectivity. (Emily had a more sophisticated argument here, but it depended on part (a), which was not presented here the same way she did it.)

We claim there can be at most 4 distinct t_i's which converge to the same point. We show this by considerng the following cases:

i. a remains in the interior of every triangle as we construct t. Then we never have more than one choice for each digit.

ii. a is in the intersection of some altitude and hypothenuse. Then a can be approached from 4 directions. But we proved in (b) that only two of these directions correspond to different t, i.e., there are two pairs and in each pair we have two representations of the *same* t.

iii. a falls on some altitude, but not on its end-point. Then we have an initial option to go to either side of this altitude. Then we can continue to choose directions as before, keeping a on the edges of our working triangle.If a never falls at the end-point of an altitude, we continue our iterations indefinitely, having only one choice for each digit. If a at some latter step does, however, fall at the end-point of an altitude, then we proceed as in the second case. This gives up to 4 distinct t_i's such that $\gamma(t_i) = a$.

□

3. Question 3

Solution. ($\Rightarrow$) Suppose f is continuousat a. Then for $\epsilon > 0$ there is a δ such that

$$|f(x) - f(a)| < \epsilon \quad \text{whenever } |x - a| < \delta.$$

For this δ there exists an N such that $|a_i - a| < \delta$ whenever $i > N$. Thus $|f(a_i) - f(a)| < \epsilon$ for $i > N$ and so $f(a_i)$ converges to $f(a)$, as claimed.

($\Leftarrow$) *(due to Simona Topor Pop)* Suppose that $\lim a_i = a \Rightarrow \lim f(a_i) = f(a)$ but that f is not continuous at a. This means there exists an $\epsilon > 0$ such that for any $\delta > 0$ there is always an x such that

$$|f(x) - f(a)| \geq \epsilon \quad \text{but } |x - a| < \delta.$$

Construct the sequence $\delta_i = 1/i$. Then we will always find x_i such that $|f(x_i) - f(a)| \geq \epsilon$. But for any $\epsilon > 0$ we can find i such that $1/i < \epsilon$, and so, by our construction, $x_i \to a$ as $i \to \infty$. But this would mean $f(x_i) \to f(a)$ as $i \to \infty$ according to our hypothesis. So there is an N such that $|f(x_i) - f(a)| < \epsilon$ for all $i > N$, contrary to what we saw before. Therefore, f must be continuous at a. □

4. Question 4

Solution. Since g is continuous at $f(a)$, for a given $\epsilon > 0$ there is a $\delta > 0$ such that $|g(y) - f(a)| < \epsilon$ whenever $|y - f(a)| < \delta$. Since f is continuous at a, we know that for $\delta > 0$ there is a $\delta' > 0$ such that $|f(x) - f(a)| < \delta$ whenever $|x - a| < \delta'$. It follows that $|(g \circ f)(x) - (g \circ f)(a)| < \epsilon$ whenever $|x - a| < \delta'$, so $g \circ f$ is continuous at a. □

5. Question 5

Remark. Many people asserted here such claims as 'between any two real numbers there is an irrational.' Such a claim is true, but it requires proof. Many people dismissed this as obvious, but it really isn't if you think about it.

Solution. (due to John Carlsson) First we'll show f is not ontinuous at any rational point c. Suppose f is continuous at some rational point c. Then for $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \quad \text{whenever } |x - c| < \delta.$$

So we need a delta that will map the interval $(c-\delta, c+\delta)$ inside the interval $(f(c)-\epsilon, f(c)+\epsilon)$. Now choose $\epsilon < f(c)$. Then our range cannot include the number zero (since $f(c) - \epsilon > 0$). In other words, we need a δ such that the interval $(c - \delta, c + \delta)$ contains no irrationals.

(This part of the proof is different from J. Carlsson's.) If δ is a rational number, then $(1/\pi)(c - \delta) + (1 - 1/\pi)(c + \delta)$ is an irrational number inside $(c - \delta, c + \delta)$. If δ is irrational, then $c + \delta/2$ is also irrational since c is rational and $c + \delta/2 \in (c - \delta, c + \delta)$. Either way we obtain a contradiction, so f is not contanuous at a rational c. (end of Tony's contribution)

Now we show that f is continuous at irrational numbers. Let c be an irrational number. Notice that $\sup\{f(x)\} = 1$. Consider any interval (a, b) in $\mathbb{R}$. Not only is $f(x)$ bounded along the interval, it also has a finite number of maxima (for instance, $f(x) = 1$ if and only if $x \in \mathbb{Z}$, and there are a finite number of integers along any interval.) Consider an interval $(k, k + 1)$ that contains c in $\mathbb{R}$. This contains at most one integer. Now partition the interval in half. Suppose, without loss of generality, that $(k + 1/2, k + 1) := I$ does not contain the integer, and that $c \in I$ (if not, shift k so that this is so). Then $\sup_{x\in I}\{f(x)\} \le 1/2$. One may keep bisecting these intervals and shifting them so that $\sup_{x\in I}\{f(x)\} \le 1/i$ for some desired i. Choose δ to be half the length of this interval so that $1/i < \epsilon$. This shows f is continuous at c. □

6. Question 6

Solution. (based on work by Arthur Baum) Consider the curve $y = f(x)$. Draw a line through two points $(u, f(u))$ and $(v, f(v))$ on the graph of f, where $u, v \in [a, b]$.

The equation of the line between these points is

$$y - f(u) = \frac{f(v) - f(u)}{v - u}(x - u).$$

Let $g\colon [u, v] \to \mathbb{R}$ map values between u and v to this line. Let's find an expression for $g(\lambda u + (1 - \lambda)v)$ where $0 \le \lambda \le 1$ and, consequently, $\lambda u + (1 - \lambda)v \in [u, v]$. We have

$$\begin{aligned}
g(\lambda u + (1-\lambda)v) - f(u) &= \frac{f(v) - f(u)}{v - u}(\lambda u + (1-\lambda)v - u) \\
[g(\lambda u + (1-\lambda)v) - f(u)] \cdot (v - u) &= [f(v) - f(u)] \cdot (\lambda u + v - \lambda v - u) \\
[g(\lambda u + (1-\lambda)v) - f(u)] \cdot (v - u) &= -[f(v) - f(u)] \cdot (v - u) \cdot (1 - \lambda) \\
[g(\lambda u + (1-\lambda)v) - f(v)] \cdot (v - u) &= [f(u) - f(v)] \cdot \lambda(v - u) \\
g(\lambda u + (1-\lambda)v) &= \lambda f(u) - \lambda f(v) + f(v) \\
g(\lambda u + (1-\lambda)v) &= \lambda f(u) + (1-\lambda) f(v)
\end{aligned}$$

Becuase f is convex, we know that $f(\lambda u + (1 - \lambda)v) \le \lambda f(u) + (1 - \lambda)f(v)$, so the value of the function at any point in $[u, v]$ lies on or below the line connecting $(u, f(u))$ and $(v, f(v))$.

Choose any $c \in (a, b)$. Plot the point $C = (c, f(c))$. Note that C lies on or below the line connecting the points $A = (a, f(a))$ and $B = (b, f(b))$. We will now draw in the lines passing through A and C and through B and C Call them l_1 and l_2 respectively. Because f is convex, we know that the image of a point in $[a, c]$ lies on or below l_1 and that the image of a point in $[c, b]$ lies on or below l_2.

We will now show by contradiction that the image of f lies on or above l_2 for points in $[a, c]$ and on or above l_1 for points in $[c, b]$. That is, we will show that f must map from $[a, b]$ to the regions bounded by lines l_1 and l_2 in the Cartesian plane between a and b. Suppose that this is not true and that a point $D = (d, f(d))$, where $d \in (a, b)$ and $a, b, c \neq d$, lies below the bounded regions. If $d \in (a, c)$ the point C will lie *above* the line BD, while if $d \in (c, b)$ the point C will lie above the line AD. However f is convex, so C should lie on or below the line between D and the appropriate endpoint.

Now we'll show that f is continuous. For $\epsilon > 0$, choose $\delta < \min\{\epsilon(c - a)/(f(c) - f(a)), \epsilon(b - c)/(f(b) - f(c))\}$. Then for $|x - c| < \delta$ we have $|l_1(x) - l_1(c)| < \epsilon$ and $|l_2(x) - l_2(c)| < \epsilon$ where $l_i(x)$ is the *function* that maps x to a number such that $(x, l_i(x))$ is on the line l_i. But this means $|f(x) - f(c)| < \epsilon$ since $f(x)$ lies between $l_1(x)$ and $l_2(x)$ and $l_1(c) = f(c) = l_2(c)$. So f is continuous at c for every $c \in [a, b]$.

Let's quickly inspect the behavior at the endpoints a and b. As long as f is continuous on (a, b), we can set $f(a)$ and $f(b)$ to any value that is at least as large as $\lim_{x \to a^+} f(x)$ or $\lim_{x \to b^-} f(x)$, respectively, and the convex relationship would still hold; that is, all points on the curve between A and C or B and C would still lie on or underneath the lines AC and BC, respectively, and the entire curve from $[a, b]$ would still lie on or beneath the line AB. So f must be continuous on the open interval (a, b), not necessarily at the endpoints.

□