MATH 25A – PROBLEM SET #4
DUE FRIDAY OCTOBER 19

1. Part A

Recall that a topological space is a pair (S, τ), where S is a set and τ a collection of subsets of S, called *open* sets, satisfying:

- $S, \emptyset \in \tau$.
- $\cup_{i \in I} U_i \in \tau$ if all $U_i \in \tau$ (here I is a possibly infinite set of indices.)
- $\cap_{i=1}^{n} U_i \in \tau$ if all $U_i \in \tau$.

A subset $C \subset S$ is called *closed* if $S \setminus C$ is open. A subset $C \subset S$ is *compact* if every open cover of C has a finite subcover. Finally, a function $f : S \to T$ between topological spaces is *continuous* if $f^{-1}(U)$ is open in S for any open $U \subset T$.

(1) Let (S, d) be a metric space. The *metric topology* on S is defined by: $U \subset S$ is open if and only if for any $a \in U$ there exists $\varepsilon > 0$ such that the open ε-ball $B_\varepsilon(a)$ lies in U.
   (a) Prove that this defines a topology.
   (b) Prove that two metrics d_1, d_2 on S are equivalent if and only if they define the same topology.

(2) Let (S, τ) be a topological space, and $T \subset S$. Define the *subset topology* on T by: $U \subset T$ is open if and only if $U = T \cap V$ for some open $V \subset S$. Prove that this defines a topology.

(3) Define a topology on $\mathbb{R}$ by: $U \subset \mathbb{R}$ is open if and only if either $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite. Prove that this defines a topology and that $\mathbb{R}$ is compact in this topology.

(4) Prove that a closed subset D of a compact set C in a topological space S is compact. (Hint: given an open cover of D, add some open sets to get a cover of C.)

2. Part B

(1) Given any topological space S, we construct a compact space by adding one point to S. This compact space is called the *one point compactification of S*. The construction is modeled after the following simple case. Let C be the unit circle in $\mathbb{R}^2$ centered at $(0, 1)$. Then lines through $P = (0, 2)$ give a one-to-one correspondence between points on the circle, different from P, and points on the x-axis. This is a continuous map with a continuous inverse, hence we may consider the circle as $\mathbb{R}$ plus one extra point P. The circle C is the one point compactification of $\mathbb{R}$.

   Let S be a topological space and $T = S \cup \{\infty\}$, where ∞ is simply a point not in S. *Assume that all compact sets in S are closed.* Define the topology on T by: $U \subset T$ is open if either U is an open set in S, or $\{\infty\} \in U$ and $S \setminus U$ is compact.
   (a) Prove that this defines a topology.
   (b) Prove that T is compact in this topology.

(2) Let $C \subset \mathbb{R}^n$ be a compact set, and $f : C \to \mathbb{R}$ a continuous map, such that $f(x) > 0$ for all $x \in C$. Prove that there exists a constant $K > 0$ such that $f(x) \geq K$ for all $x \in C$.

(3) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, such that $Ker(L) = 0$. Show that there exists a constant $K > 0$ such that

$$|L(v)| \geq K|v|$$

for all $v \in \mathbb{R}^n$. (Hint: first find a K that works for all $|v| = 1$.)

(4) Let $C \subset \mathbb{R}^n$ be a compact set, and $f : C \to \mathbb{R}^m$ a continuous injective map. Because f is injective, one can define the inverse map $f^{-1} : f(C) \to C$. Prove that f^{-1} is continuous. (Hint: The problem becomes easy once you choose the right definition of continuity.)

3. Part C

Let $S \subset \mathbb{R}^n$. We define the *boundary* of S, $bd(S)$ as the set of $x \in \mathbb{R}^n$ such that for any $\varepsilon > 0$ the open ball $B_\varepsilon(x)$ contains points from S and from $\mathbb{R}^n \setminus S$. We define the *closure* of S, $cl(S)$ to be the intersection of all closed sets in $\mathbb{R}^n$ containing S. The *interior* of S, $int(S)$ is defined to be the union of all open sets contained in S.

(1) Prove that $S \subset \mathbb{R}^n$ is closed if and only if $bd(S) \subset S$.
(2) Prove that $S \cup bd(S)$ is closed.
(3) Prove that $cl(S)$ is a closed set and $cl(S) = S \cup bd(S)$.
(4) Prove that $int(S)$ is an open set and $int(S) = S \setminus bd(S)$.
(5) Construct the Cantor set as follows. Start with the interval $[0, 1]$. Remove the middle third $(1/3, 2/3)$, then remove the middle thirds of each of the two remaining intervals, and so on. Is Cantor set closed ? What is its boundary ?