Problem Set #4a – Solutions

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Math 25

Note 1: A disturbing number of people defined a compact set as being closed and bounded. *THIS IS NOT TRUE IN AN ARBITRARY TOPOLOGICAL SPACE.* What we have proven is that if $U \subset \mathbb{R}^n$ is closed *under the usual topology* and bounded *under the usual Euclidean metric*, then it is compact *under the usual topology.* In general, though, sets are compact iff every open cover has a finite subcover. In many spaces, this is not equivalent to being closed and bounded! In fact, unless the space we are dealing with is a metric space, the notion of "bounded" really doesn't have any meaning. For example, consider $S := \{\text{all subsets of } \mathbb{Z}\}$. We can certainly define the cofinite topology (as in problem 3) on this space, but we haven't defined any metric on S, so we don't really know what it means for a subset of S to be bounded.

Furthermore, even if we restrict ourselves to metric space under their induced metric topologies, it is still possible for a set to be, for e xample, closed and bounded but not compact. You can convince yourself of this by considering the metric

$$d(x, y) := Min(|x - y|, 1)$$

on $\mathbb{R}$. You can verify that this metric induces the usual topology, so we know that $\mathbb{R}$ is closed but not compact, yet it is bounded under the metric d.

Note 2: A fair number of people assumed that an arbitrary (or, more precisely, set-indexed) union (or intersection) was a countable union (or intersection). As many of you may not be aware that you did this, here is an example:

$$\cup_{i \in I} U_i = U_{i_1} \cup U_{i_2} \cup \dots$$

This is not always true. Suppose $I = \mathbb{R}$, then there is no sequence $i_1, i_2, \dots$ that includes all elements of I. (This is because $Card(\mathbb{R}) = \aleph_1$ whereas $Card(\mathbb{Z}) = \aleph_0$.)

Note 3: Many people also made mistakes (particularly in number 4) regarding unions of covers. Remember that an open cover is a set of *sets*, whereas an open set is a set of *points*. Suppose $\mathfrak{U} := \{U_1, U_2, \dots\}$ is a (countable) open cover of some set S, and we want to add another open set, $V \subset S$ to it. It makes a lot of sense to consider

$$\begin{aligned} \mathfrak{U}' &:= \mathfrak{U} \cup \{V\} \\ &= \{U_1, U_2, \dots\} \cup \{V\} \\ &= \{V, U_1, U_2, \dots\} \end{aligned}$$

(Note that we should almost certainly say that $V \in \mathfrak{U}'$ and not that $V \subset \mathfrak{U}'$.) On the other hand, suppose that we omitted the curly braces in the first line of the above. (Let's say that $V = \{p_1, p_2, \dots\}$, a countable set.) Then we would have:

$$\begin{aligned} \mathfrak{U}'' &:= \mathfrak{U} \cup V \\ &= \{U_1, U_2, \dots\} \cup \{p_1, p_2, \dots\} \\ &= \{U_1, U_2, \dots, p_1, p_2, \dots\} \end{aligned}$$

In $\mathfrak{U}''$, we have a mixture of open sets and single points. While this is well-defined as a set, it is not a cover.

You may also find the following notation useful. If $\mathfrak{U}$ is a set of sets, then $\cup\mathfrak{U}$ is the union of all elements of $\mathfrak{U}$. For example, if $\mathfrak{U}$ is the finite collection $\{U_1, U_2, \ldots, U_n\}$, then $\cup\mathfrak{U} = U_1 \cup U_2 \cup \ldots \cup U_n$. Similarly, $\cap\mathfrak{U}$ is the intersection of all elements of $\mathfrak{U}$.

Problem 1.

Part i.

Claim 1.1. (S, τ) is a topology.

We must show that the three axioms for a topology are satisfied by (S, τ):

$S, \varnothing \in \tau$: We first note that $\varnothing$ vacuously satisfies the criterion for openness. We next note that, by definition, $B_\epsilon(a)$ is always a subset of S. Thus, we can select any $\epsilon > 0$ we would like, say 1, and we have that

$$\forall a \in S,\ B_\epsilon(a) \subset S$$

Thus, both $\varnothing$ and S are elements of τ.

$U_i \in \tau \Rightarrow \cap_{i \in I} U_i \in \tau$: Let's define $U := \cup_{i \in I} U_i$. We must prove that $U \in \tau$. If $U = \varnothing$ then we are done (by the previous axiom). Otherwise, let a be an arbitrary element of U. We must show that $\exists \epsilon$ such that $B_\epsilon(a) \subset U$. We know that $a \in U_j$ for some $j \in I$. By the openness of U_j, we know $\exists \epsilon$ such that $B_\epsilon(a) \subset U_j \subset U$. Because our choice of a was arbitrary, we have shown that U is open.

$U_i \in \tau \Rightarrow \cap_{i=1}^n U_i \in \tau$: Let's define $U := \cap_{i=1}^n U_i \in \tau$. We must prove that $U \in \tau$. If $U = \varnothing$ then we are done (by the previous axiom). Otherwise, let a be an arbitrary element of U. We must show that $\exists \epsilon$ such that $B_\epsilon(a) \subset U$. We know that $\forall j \in \{1, \ldots, n\}, a \in U_j$. By the openness of each U_j, we know that there exists $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that

$$\forall i, B_{\epsilon_i}(a) \subset U_i$$

Because any finite set has a minimal element, we can define $\epsilon = min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Thus $\forall i, \epsilon \leq \epsilon_i$, so $B_\epsilon(a) \subset B_{\epsilon_i}(a) \subset U_i$. Thus $B_\epsilon(a) \subset U$ and the claim is proven.

□

Part ii.

Claim 1.2. Two metrics, d_1 and d_2 on S, are equivalent iff they induce the same metric topology on s.

Define τ_1 to be the metric topology induced by d_1 and τ_2 to be the metric topology induced gy d_2. We will use the following notation for open balls:

$$B^1_\epsilon(a) := \{p \mid d_1(a, p) < \epsilon\}$$

$$B^2_\epsilon(a) := \{p \mid d_2(a, p) < \epsilon\}$$

$\Rightarrow$. (Assume $d_1 \sim d_2$ in order to show $\tau_1 = \tau_2$.)

Let U be an arbitrary element of τ_1. We will show that it is also in τ_2. Because $U \in \tau_1$, we know that for all $a \in U$, $\exists \epsilon_1 > 0$ such that $B^1_{\epsilon_1}(a) \subset U$. But, by the equivalence of d_1 and d_2, we know that $\exists \epsilon_2 > 0$ such that

$$B^2_{\epsilon_2}(a) \subset B^1_{\epsilon_1}(a) \subset U$$

Thus, U is open under τ_2. By symmetry (i.e. by swapping 1's and 2's in the preceding paragraph), we also know that $U \in \tau_2 \Rightarrow U \in \tau_1$. Thus $\tau_1 = \tau_2$.

$\Leftarrow$. (Assume $\tau_1 = \tau_2$ in order to show that $d_1 \sim d_2$.)

We first note that for any $\epsilon_1 > 0$, $B^1_{\epsilon_1}(a)$ is open in $\tau 1$. Specifically, for any $p \in B^1_{\epsilon_1}$, we select $\epsilon' := \epsilon_1 - d_1(p,a)$ (which must be positive because $p \in B^1_{\epsilon_1}(a) \Rightarrow d(p,a) < \epsilon_1$). Then

$$B^1_{\epsilon'}(p) \subset B^1_{\epsilon_1}(a)$$

Because $B^1_{\epsilon_1}(a)$ is open in τ_1, it must also be open in τ_2. Thus, for all $p \in B^1_{\epsilon_1}(a)$, $\exists \epsilon_2$ such that $B^2_{\epsilon_2}(p) \subset B^1_{\epsilon_1}(a)$. We can select p to be a, and we have shown that $\forall a, \epsilon_1$, $\exists \epsilon_2$ such that

$$B^2_{\epsilon_2}(a) \subset B^1_{\epsilon_1}(a)$$

By symmetry, we can have also shown that $\forall a, \epsilon_2$, $\exists \epsilon_1$ such that

$$B^1_{\epsilon_1}(a) \subset B^2_{\epsilon_2}(a)$$

and have thereby established the equivalence of d_1 and d_2.

□

Problem 2.

Claim 2.1. The subspace topology is actually a topology.

Let's name the subspace topology on T σ. Thus the claim is equivalent to verifying the three axioms of topological spaces for (T, σ)

$\varnothing, T \in \sigma$: We know that $\varnothing \in \tau$, so $\varnothing \cap T = \varnothing \in \sigma$. Likewise, because $S \in \tau$, $S \cap T = T \in \sigma$.

$U_i \in \sigma \to \cup_{i\in I} U_i \in \sigma$: For all $i \in I$, we know that $U_i \in \sigma$. Thus $\exists V_i \in \tau$ such that $U_i = T \cap V_i$. Then

$$\begin{aligned} \cup_{i\in I} U_i &= \cup_{i\in I}(T \cap V_i) \\ &= T \cap (\cup_{i\in I} V_i) \end{aligned}$$

Because $V_i \in tau$ and τ is a topology, we know that $\cup_{i\in I} V_i \in \tau$ thus $\cup_{i\in I} U_i \in \sigma$.

$U_i \in \sigma \Rightarrow \cap_{i=1}^n U_i \in \sigma$: For all $i \in [1, \ldots, n]$, e know that $U_i \in \sigma$. Thus $\exists V_i \in \tau$ such that $U_i = T \cap V_i$. Then

$$\begin{aligned} \cap_{i=1}^n U_i &= \cap_{i=1}^n (T \cap V_i) \\ &= T \cap (\cap_{i=1}^n V_i) \end{aligned}$$

Because $V_i \in tau$ and τ is a topology, we know that $\cap_{i=1}^n V_i \in \tau$ thus $\cap_{i=1}^n U_i \in \sigma$.

□

Problem 3.

Part i.

Claim 3.1. The cofinite topology (defined by the property that non-empty open sets are the complement of finitely many points) is a topology on $\mathbb{R}$.

Let's define $\tau = \{U \subset \mathbb{R} \mid \mathbb{R} \setminus U \text{ is finite}\} \cup \{\varnothing\}$. Then we must prove that $(\mathbb{R}, \tau)$ satisfies the axsioms of a topological space.

$\varnothing, \mathbb{R} \in \tau$: $\varnothing$ is explicitly included in τ. For $\mathbb{R}$, we note that $\mathbb{R} \setminus \mathbb{R} = \varnothing$ which is finite.

$U_i \in \tau \Rightarrow \cup_{i\in I} U_i \in \tau$: For all $i \in I$, we know that $U_i \in \tau$. Thus $V_i := \mathbb{R} \setminus U_i$ is finite. Consequently, we have:

$$\begin{aligned} \cup_{i\in I} U_i &= \cup i \in I(\mathbb{R} \setminus V_i) \\ &= \mathbb{R} \setminus \cap_{i\in I} V_i \end{aligned}$$

But, as the intersection of finite sets, $\cap_{i\in I} V_i$ is finite, thus $\cup_{i\in I} U_i \in \tau$.

$U_i \in \tau \Rightarrow \cap_{i=1}^n U_i \in \tau$: For all $i \in [1, \dots, n]$, we know that $U_i \in \tau$. Thus $V_i := \mathbb{R} \setminus U_i$ is finite. Consequently, we have:

$$\begin{aligned} \cap_{i=1}^n U_i &= \cap i = 1^n(\mathbb{R} \setminus V_i) \\ &= \mathbb{R} \setminus \cup_{i=1}^n V_i \end{aligned}$$

But, as the finite union of finite sets, $\cup_{i=1}^n V_i$ is finite, thus $\cap_{i=1}^n U_i \in \tau$.

□

Part ii.

Claim 3.2. $\mathbb{R}$ is compact under the cofinite topology.

Let an open cover, $\mathfrak{U}$, of $\mathbb{R}$ be given. We will endeavor to prove that $\mathfrak{U}$ contains a finite subcover. Let U be a non-empty element of $\mathfrak{U}$. (There must be some non-empty element of U, or else we would conclude that $\cup\mathfrak{U} = \varnothing \not\supset \mathbb{R}$ which contradicts the assumption that $\mathfrak{U}$ is a cover of $\mathbb{R}$.) Then we know that $\mathbb{R} \setminus U = \{p_1, \dots p_n\}$. But for all $j \in [1, \dots, n]$, $p_j \in \mathbb{R} \subset \cup\mathfrak{U}$, so $\exists U_j \in \mathfrak{U}$ such that $p_j \in U_j$. Then take all of these U_j's – along with U – for the finite subcover. That is, define

$$\mathfrak{V} := \{U\} \cup (\cup_{i=1}^n \{U_i\})$$

Then $\mathfrak{V}$ is a finite subcover of $\mathbb{R}$ by elements of $\mathfrak{U}$.

□

Problem 4.

Claim 4.1. Let (S, τ) be a topological space. Let $C \subset S$ be a compact set (under τ), and let $D \subset C$ be a closed subset of S. Then D is compact.

Let $\mathfrak{U}$ be an open cover of D. We must show that it contains a finite subcover. We know that D is closed. Thus its complement, $S \setminus D$, is open. (**Note that $C \setminus D$ is not (necessarily) the complement of D. In fact, $C \setminus D$ isn't even necessarily open.** Consider, for example, $S = \mathbb{R}$, $\tau =$ the usual (i.e. Euclidean metric) topology on $\mathbb{R}$, $C = [0, 1]$, and $D = \left[0, \frac{1}{2}\right]$. Then $C \setminus D = \left(\frac{1}{2}, 1\right]$ which definitely isn't open!) Thus we can define an open cover $\mathfrak{V} := \mathfrak{U} \cup \{S \setminus D\}$. We note that

$$\begin{aligned} \cup\mathfrak{V} &= (\cup\mathfrak{U}) \cup (S \setminus D) \\ &\supset D \cup (S \setminus D) \\ &= S \\ &\supset C \end{aligned}$$

Thus $\mathfrak{V}$ is an open cover of C, so by the compactness of C, it has a finite subcover, $\mathfrak{V}'$. Because $D \subset C$, we know that $\mathfrak{V}'$ must be a finite open cover of D. We now consider two cases:

Case i: $\mathfrak{V}' \subset \mathfrak{U}$. In this case, $\mathfrak{V}'$ is our open subcover and we are done.

Case ii: $\mathfrak{V}' \not\subset \mathfrak{U}$. We know that $\mathfrak{V}' \subset \mathfrak{V}$ and $\mathfrak{V} \setminus \mathfrak{U} = \{S \setminus D\}$. Intuitively, we can therefore remove $(S \setminus D)$ from $\mathfrak{V}'$ and be sure to be left with a finite subcover. Formally, we define

$\mathfrak{U}' := \mathfrak{V}' \setminus \{S \setminus D\}$. We then note that $\mathfrak{U}'$ is a finite subset of $\mathfrak{U}$ and observe that

$$\forall p \in D, p \in \cup\mathfrak{V}' \Rightarrow (p \in \cup\mathfrak{U}' \text{ or } p \in (S \setminus D))$$

But the latter is not possible (as $p \in D$), thus we infer that $\cup\mathfrak{U}' \supset (\cup\mathfrak{V}') \cap D$ and affirm that $\mathfrak{U}'$ is a finite subcover of D by elements of $\mathfrak{U}$.

□