Problem Set 4, Part B Solution Set

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1. Given any topological space S, we construct a compact space by adding one point to S. This compact space is called the *one point compactification of S*.

   Let S be a topological space and $T = S \cup \{\infty\}$, where ∞ is simply a point not in S. *Assume that all compact sets in S are closed.* Define the topology on T by: $U \subset T$ is open if either U is an open set in S, or $\{\infty\} \in U$ and $S \smallsetminus U$ is compact.

   (a) Prove that this defines a topology.

   Solution. Let τ be the topology associated to S, and τ' be the collection of open sets defined above. To prove τ' is a topology, we need to show that $\emptyset$ and T are open sets, that arbitrary (not necessarily countable) unions of open sets are open, and that finite intersections are open.

   - $\emptyset$ and T are open sets. We know $\emptyset \in \tau$, *i.e.*, $\emptyset$ is open in S, so $\emptyset \in \tau'$. Since $\infty \in T$, T is an open set in τ' if $S \smallsetminus T$ is compact. But $S \smallsetminus T = \emptyset$, which is compact (it satisfies the definition vacuously).
   - Arbitrary unions of sets in τ' are in τ'. Let $\{U_i\}_{i \in I}$ be an arbitrary collection of open sets in τ' We consider three cases:
     i. $\infty \notin U_i$ for all $i \in I$. Then all the U_i are open in S, and since τ is a topology, $\cup_{i \in I} U_i \in \tau$, in other words, $\cup_{i \in I} U_i$ is open in S, so this union is in τ'.
     ii. $\infty \in U_i$ for all $i \in I$. Then $\infty \in \cup_{i \in I} U_i$. In this case, the arbitrary union will be open if
     $$S \smallsetminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} (S \smallsetminus U_i) \tag{1}$$
     is compact. We know that each $S \smallsetminus U_i$ is compact since $\infty \in U_i$. We have assumed that compact sets are closed (see Remark below), so each $S \smallsetminus U_i$ is closed in S. Since τ is a topology, we have that their arbitrary intersection is also a closed set in S. Hence the left hand side of (1) is a closed set in S. But we also have $\cap_{i \in I}(S \smallsetminus U_i) \in S \smallsetminus U_i$ for any i. So $\cap_{i \in I}(S \smallsetminus U_i)$ is a closed subset of a compact set, and by problem A4, it is compact itself.
     iii. We get a mixed bag. Then $\infty \in U_i$ for some i. then write
     $$\bigcup_{i \in I} U_i = \bigcup_{i \in J} U_j \cup \bigcup_{i \in K} U_k$$

where $J \subset I$ such that $\infty \notin U_j$ for $j \in J$ and $K \subset I$ such that $\infty \in U_k$ for $k \in K$. But then $\cup_j U_j$ is open by case (i) and $\cup_k U_k$ is open by case (ii).

Thus it suffices to prove that $U \cup V$ is open in τ' for open sets U, V such that $\infty \notin U$ but $\infty \in V$. Since $\infty \in U \cup V$, we need to show that $S \smallsetminus (U \cup V)$ is compact. But $S \smallsetminus (U \cup V) = (S \smallsetminus U) \cap (S \smallsetminus V)$. We know that $S \smallsetminus U$ is closed in S since U is open, and $S \smallsetminus V$ is closed in S since it is compact (again, see the remark below). So the intersection $(S \smallsetminus U) \cap (S \smallsetminus V)$ is closed in S. And $S \smallsetminus (U \cup V) \subset S \smallsetminus V$, which is compact, so by problem A4, $S \smallsetminus (U \cup V)$ is compact.

- finite intersections of elements in τ' are in τ'. Let $\{U_i\}$, $i = 1, \dots, n$ be a finite collection of open sets in τ' We consider three cases:

  i. $\infty \notin U_i$ for all $i = 1, \dots n$. Then all the U_i are open in S, and since τ is a topology, $\cap_1^n U_i \in \tau$, in other words, $\cap_1^n U_i$ is open in S, so this intersection is in τ'.

  ii. $\infty \in U_i$ for all $i = 1, \dots n$. Then $\infty \in \cap_1^n U_i$. In this case, the finite intersection will be open if

$$S \smallsetminus \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (S \smallsetminus U_i) \tag{2}$$

  is compact. Again, we know that each $S \smallsetminus U_i$ is compact (and therefore closed) since $\infty \in U_i$. Since τ is a topology, we have that the finite union of closed sets is also a closed set in S. Hence the left hand side of (2) is a closed set in S. But we also have $\cup_1^n (S \smallsetminus U_i) \subset S \smallsetminus U_i$ for any i. So $\cup_1^n (S \smallsetminus U_i)$ is a closed subset of a compact set, and by problem A4, it is compact itself.

  iii. We get a mixed bag. Then $\infty \in U_i$ for some i. then write

$$\bigcap_{i=1}^{n} U_i = \bigcap_{i \in J} U_j \cap \bigcap_{i \in K} U_k$$

  where $J \subset I = \{1, \dots, n\}$ such that $\infty \notin U_j$ for $j \in J$ and $K \subset I$ such that $\infty \in U_k$ for $k \in K$. But then $\cap_j U_j$ is open by case (i) and $\cap_k U_k$ is open by case (ii).

  Like with arbitrary unions, it suffices to prove that $U \cap V$ is open in τ' for open sets U, V such that $\infty \notin U$ but $\infty \in V$. V is a set of the form $T \smallsetminus C$ where C is a compact set of S. Thus $U \cap V = U \cap (T \smallsetminus C) = U \cap (S \smallsetminus C)$ since $\infty \notin U$. But U is open in S, and so is $S \smallsetminus C$, because C is closed (it is compact). So $U \cap (S \smallsetminus C)$ is open in S, and therefore it is a member of τ'.

□

(b) Prove that T is compact in this topology.

Solution. Let $\mathcal{A}$ be an open cover of T. Then there is a $U \in \mathcal{A}$ such that $\infty \in U$ (otherwise $\mathcal{A}$ would not cover T). We know $S \smallsetminus U$ is compact for this U, and $\mathcal{B} = \mathcal{A} \smallsetminus \{U\}$ covers $S \smallsetminus U$. Hence there is a finite subcover of $\{U_{i_1}, \dots, U_{i_n}\} \subset \mathcal{B}$ of $S \smallsetminus U$. Then $\{U_{i_1}, \dots, U_{i_n}, U\}$ is a finite subcover ot T. Hence T is compact. □

Remark. We used the assumption that compact sets are closed. This is not true in general topological spaces. It is only true in spaces that are *Hausdorff.* A topological space X is said to be Hausdorff it for any two points $x_1, x_2 \in X$ there are disjoint open sets U and V that contain x_1 and x_2 respectively.

Theorem 1. *A compact subspace of a Hausdorff space is closed.*

Proof. We'll follow Munkres (*Topology*, second edition, p 165) in our proof. Let Y be a compact subspace of the Hausdorff space X. We'll show $X \smallsetminus Y$ is open. Let x be a point in $X \smallsetminus Y$. We'll show there is an open set containing x which is entirely contained in $X \smallsetminus Y$. For each $y \in Y$, choose disjoint open sets U_y and V_y of x and y respectively. We can do this because our space is Hausdorff. Then $\{V_y | y \in Y\}$ is an open cover of Y. Since Y is compact, there is a finite subcover $\{V_{y_1}, \ldots, V_{y_n}\}$. Take $V = V_{y_1} \cup \cdots \cup V_{y_n}$. This is an open set that contains Y, and it is dijoint from the open set $U = U_{y_1} \cap \cdots \cap U_{y_n}$. So U is an open set containing x completely contained in $X \smallsetminus Y$. □

2. Let $C \subset \mathbb{R}^n$ be a compact set, and $f : C \to \mathbb{R}$ a continuous map, such that $f(x) > 0$ for all $x \in C$. Prove that there exists a constant $K > 0$ such that $f(x) \geq K$ for all $x \in C$.

   Since C is a compact set, and f is a continuous function, then f achieves its minimum on C. That is, there is a point $a \in C$ for which $f(x) \geq f(a)$ for all $x \in C$. Furthermore, $f(a) > 0$ by hypothesis. So just take $K = f(a)$.

3. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, such that $Ker(L) = 0$. Show that there exists a constant $K > 0$ such that

$$|L(v)| \geq K|v|$$

   for all $v \in \mathbb{R}^n$. (Hint: first find a K that works for all $|v| = 1$.)

   Solution. A linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, and therefore continuous by what was shown in class. Note also that $|L(v)| : \mathbb{R}^n \to \mathbb{R}$ is continuous as it is the composition of L and the magnitude function, both of which are continuous.

   Now consider the set of vectors $v \in \mathbb{R}^n$ of unit length. They lie in the unit sphere S^{n-1}, which is compact since it is closed and bounded. Since $\ker L = \{0\}$ and all $v \in S^{n-1}$ are non-zero, we have $|L(v)| > 0$. So now we apply the result from problem 2. That is, there exists a $K > 0$ for which $|L(v)| > K$. Now consider *any* vector in $v \in \mathbb{R}^n$. We have, using the linearity of L,

$$|L(v)| = \left|L\left(v \cdot \frac{|v|}{|v|}\right)\right| = |v| \cdot \left|L\left(\frac{v}{|v|}\right)\right| > K \cdot |v|.$$

□

4. Let $C \subset \mathbb{R}^n$ be a compact set, and $f : C \to \mathbb{R}^m$ a continuous injective map. Because f is injective, one can define the inverse map $f^{-1} : f(C) \to C$. Prove that f^{-1} is continuous.

   Remark. Many people are not clear on what the open/closed sets definition of continuity means. If a function f is continuous, this *does not mean* f maps open set to open sets. Rather, it means that f *pulls back* open sets in the range to open sets in the domain, and similarly for closed sets.

Solution (based on work by Kevin Weil). Let D be a closed subset of C. Since C is compact, D is compact as well (problem A4). To show f^{-1} is continuous we need $(f^{-1})^{-1}(D)$ to be closed. Since f is inyective, f^{-1} is well-defined. What is *not obvious* is that $(f^{-1})^{-1} = f$. First, convince yourself f^{-1} is inyective. Then inyectivity of *both* f and f^{-1} gives $(f^{-1})^{-1} = f$.

Now that we know $(f^{-1})^{-1} = f$, we obtain $(f^{-1})^{-1}(D) = f(D)$. Since D is compact, $f(D)$ is a compact subset of $\mathbb{R}^m$, and is therefore closed. So f^{-1} is continuous. $\square$