Math 25a Solution Set #4 (Part A)

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Let $S \subset \mathbb{R}^n$. We define the boundary of S, $bd(S)$ as the set of $x \in \mathbb{R}^n$ such that for any $\epsilon > 0$ the open ball $B_\epsilon(x)$ contains points from S and from $\mathbb{R}^n - S$.

We define the closure of S, $cl(S)$ to be the intersection of all closed sets in $\mathbb{R}^n$ containing S.

The interior of S, $int(S)$ is defined to be the union of all open sets contained in S.

NOTE: Throughout this problem set, students are urged to keep in mind that there exist sets which are neither closed nor open; likewise, there exist sets which are both open and closed. Hence, to show a set is open, assuming that it is closed and getting a contradiction won't work, because "not open"$\neq$"closed" !!! You can, however, show that a set is open by showing that its complement is closed. The proper form a proof by contradiction that attempts to show a set is open would start with assuming the negation of the definition of "open" holds, so somewhat like this: "We want to show S is open. So assume S is not open, then there exists a point $x \in S$ such that for all $\epsilon > 0$, $B_\epsilon(x)$ contains a point outside of S..." or alternatively: "We want to show S is open. It is enough to show S^c is closed. Assume, for the sake of contradiction, that S^c is not closed. Then there exists a sequence of elements of S^c which..."

ANOTHER NOTE: Some basic set-theory seemed to trouble a lot of people on this part of the homework. When we say that "X is not a subset of Y", that doesn't imply that $X \cap Y = \emptyset$ or that $X \subset Y^c$. It just means there are some $x \in X$ such that $x \notin Y$, but the possibility that there are some (other) $x \in X$ for which $x \in Y$ so that $x \in X \cap Y \neq \emptyset$ is not excluded. In general, it is best to use quantifiers when stating and negating set-theoretic claims, otherwise you can easily end up showing something you don't really want to show, the professor didn't really ask you to show, and/or your CA really doesn't want to read and grade...

YET ANOTHER NOTE: Same goes for statements about boundaries, closures and interiors. When talking about neighbourhoods of a point, it is crucial to specify what neihgbourhood(s) you are talking about. If you just make a claim about $B_\epsilon(x)$, out of the blue, without mentioning what x and ϵ are, that could mean LOTS of things; It can mean that you are talking about:

(1) all possible ϵ-balls around some fixed point x.

(2) for some ϵ mentioned before, ϵ-balls around all points $x \in X$, where X is some set.

(3) a single ϵ-ball around some fixed point x.

(4) all ϵ-balls arround all x in some set X...

Since most of the questions regarding boundaries, closures and interiors can be reduced to showing some properties of some OR all ϵ-balls around some OR all points in certain sets, using quantifiers was crucial here as well, since the possibility for confusion was in abundance as the example above shows... The choice between using plain English (as in "there exists", "for all" etc.) or appropriate mathematical notation is up to you.

Problem 1

Prove that $S \subset \mathbb{R}^n$ is closed if and only if $bd(S) \subset S$.

$\rightarrow$ Let $S \subset \mathbb{R}^n$ be closed, and consider any $x \in bd(S)$. We want to show $x \in S$. We do that using proof by contradiction. Assume that $x \notin S$, so that $x \in S^c$, the complement of S. Since S is closed, S^c is open, so that there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset S^c$. But then this neighbourhood of x contains no points from S, which, given the definition of $bd(S)$ above, contradicts our choice of x in $bd(S)$. We conclude that $x \in bd(S) \rightarrow x \in S$, so that $bd(S) \subset S$.

$\leftarrow$ Let $S \subset \mathbb{R}^n$ be such that $bd(S) \subset S$. We want to show S is closed and will do so by showing S^c is open. Again, for the sake of contradiction, suppose S^c isn't open. Then there exists $x_0 \in S^c$ such that for all $\epsilon > 0$, $B_\epsilon(x_0)$ does not lie entirely in S^c. This means that for all $\epsilon > 0$, $B_\epsilon(x_0) \cap S \neq \emptyset$. We also know $x_0 \in B_\epsilon(x_0)$ for all $\epsilon > 0$, and $x_0 \in S^c$, so that for all $\epsilon > 0$, $B_\epsilon(x_0) \cap S^c \neq \emptyset$. Then, by the definition of $bd(S)$, $x_0 \in bd(S)$. We know, however, that $bd(S) \subset S$, so we conclude $x_0 \in S$. This contradicts our choice of x_0 as a point in S^c. Hence, our assumption that S^c isn't open must be false. We conclude that S^c is open, so that S is closed, and we are done. ∎

Problem 2

Prove that $S \cup bd(S)$ is closed for all $S \subset \mathbb{R}^n$.

Let $A = S \cup bd(S)$. We want to show that A is closed, and will do so by showing $bd(A) \subset A$, which should suffice by the previous problem.

(NOTE that this problem isn't trivial (Not kidding!), since we don't know automatically that $bd(S \cup bd(S)) = bd(S)$. We do know $bd(S) \subset S \cup bd(S) = A$, but this per se doesn't let us use problem (1) to conclude A is closed unless we show $bd(A) \subset bd(S)$ beforehand, which could be the main chunk of this problem, if you set out to do it that way.)

So fix some $x_0 \in bd(A)$. We shall show that $x_0 \in bd(S)$, by showing that for all $\epsilon > 0$, $B_\epsilon(x_0)$ contains points both in S and in S^c. Fix some $\epsilon > 0$.

Since $x \in bd(A)$, we know $B_\epsilon(x_0) \cap A^c \neq \emptyset$. But $S \subset A \rightarrow A^c \subset S^c$, so that $B_\epsilon(x_0) \cap A^c \neq \emptyset \rightarrow B_\epsilon(x_0) \cap S^c \neq \emptyset$.

Moreover, $x \in bd(A)$ also gives us that $B_\epsilon(x_0) \cap A \neq \emptyset$, so that $B_\epsilon(x_0)$

contains a point of either S or $bd(S)$ or both. But if $B_\epsilon(x_0)$ contains a point of $bd(S)$, say $y_0 \in B_\epsilon(x_0) \cap bd(S)$, we have that any neighbourhood of y_0 contains a point of S. So we pick that neighbourhood small enough so that it is entirely contained in $B_\epsilon(x_0)$ - in other words, we pick $\delta > 0$ so that $B_\delta(y_0) \subset B_\epsilon(x_0)$, and we can do that since $y_0 \in B_\epsilon(x_0)$ and $B_\epsilon(x_0)$ is open. Now the condition that $B_\delta(y_0)$ contains a point of S implies that $B_\epsilon(x_0)$ contains a point of S.

The preceding two paragraphs show that our $B_\epsilon(x_0)$ contains points both in S and in S^c. Moreover, this holds for all ϵ, so that $x_0 \in bd(S)$. This last conclusion holds for all $x_0 \in bd(A)$, so we conclude $bd(A) \subset bd(S) \subset bd(S) \cup S = A$, and the previous problem allows us to conclude A is closed. ∎

Problem 3

Prove that $cl(S)$ is a closed set and $cl(S) = bd(S) \cup S$.

$cl(S)$ is defined as some intersection of closed sets, so, by the alternative definition of a topological space in terms of closed sets, it is closed itself.

$A = bd(S) \cup S$ is a closed set by the previous problem. As a closed set containing S, A is one of the sets whose intersection is taken in order to obtain $cl(S)$, so we conclude that $cl(S) \subset A$, and it is left to show $A \subset cl(S)$.

So fix any $x \in A = S \cup bd(S)$, and suppose, for the sake of contradiction that $x \notin cl(S)$. Since $S \subset cl(S)$, we must have $x \notin S$, so that $x \in A$ implies $x \in bd(S)$. Now, since $x \in (cl(S))^c$, which is open since $cl(S)$ is closed, we let $\epsilon > 0$ be such that $B_\epsilon(x) \subset (cl(S))^c \subset S^c$. But this means $B_\epsilon(x)$ contains no points of S, which contradicts $x \in bd(S)$. Hence our assumption $x \notin cl(S)$ is false, and we conclude that $A \subset cl(S)$ and hence $A = cl(S)$. ∎

Problem 4

Prove that $int(S)$ is an open set and $int(S) = S - bd(S)$.

$int(S)$ is defined as some union of sets all of which are open, so that it is open as a union of open sets.

Moreover, keeping in mind that "A is open" is equivalent to "$A^c = \mathbb{R}^n - A$ is closed", and using DeMorgan's laws, we get:

$$\mathbb{R}^n - int(S) = \mathbb{R}^n - \bigcup_{A \subset S} A = \bigcap_{A \subset S} A^c$$

where the union is taken over all open subsets A of S. As A goes over all open subsets of S, A^c goes over all closed sets that contain S^c, so the above gives us that $(int(S))^c = cl(S^c)$. Using the previous problem and the property of the boundary that $bd(S) = bd(S^c)$ which is obvious from its defintion, we get that $cl(S^c) = S^c \cup bd(S^c) = S^c \cup bd(S)$, so that $int(S) = (cl(S^c))^c = (S^c \cup bd(S))^c = S \cap (bd(S))^c = S - bd(S)$, with next to the last step following from DeMorgan's laws. ∎

Problem 5

Construct the Cantor set as follows. Start with the interval $[0,1]$. Remove the middle third $(\frac{1}{3}, \frac{2}{3})$, then remove the middle thirds of each of the two remaining intervals, and so on. Is Cantor set closed? What is its boundary?

Let C be the Cantor set, and C_n the set obtained after n iterations of the above described procedure. The complement of C is a union of open intervals which were discarded at each step, so as a union of open intervals it is open itself. Cantor set is thus a complement of an open set, so closed.

We shall show that Cantor set is its own boundary, i.e. that $C = bd(C)$. Since C is closed, we know that $bd(C) \subset C$, so it is only left to show $C \subset bd(C)$. In order to do that, we fix some $x_0 \in C$ and $\epsilon > 0$ and show that $B_\epsilon(x_0)$ contains some point in the complement of C.

To do that, just pick $n \in \mathbb{N}$ large enough so that $\frac{1}{3^n} < \epsilon$. Since after the n-th iteration C_n consists of intervals all of whom have length $\frac{1}{3^n}$, and $(x - \frac{1}{3^n}, x + \frac{1}{3^n})$ has length $\frac{2}{3^n}$, $B_{\frac{1}{3^n}}(x_0)$ and hence $B_\epsilon(x_0)$ must contain a point outside of C_n and hence ouside of C. ∎