Problem Set #5b – Solutions

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Problem 1.

Claim 1.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be homogeneous of degree m (i.e. $f(kx) = k^m f(x)$) and have continuous partial derivatives. Then

$$\sum_{i=1}^{n} x_i D_i f(x) = m f(x)$$

Using the fact that $D_i f(x) := Df(x)(\vec{e_i})$, we can rewrite the left hand side. We then use linearity of $Df(x)$ and the fact that the partial derivatives of f are continuous:

$$\begin{aligned} \sum_{i=1}^{n} x_i Df(x) &= \sum_{i=1}^{n} x_i Df(x)(\vec{e_i}) \\ &= Df(x)(x) \\ &= D_x f(x) \end{aligned}$$

We now continue, using the definition of a partial derivative in the x direction:

$$\begin{aligned} &= \lim_{h\to 0} \frac{f(\vec{x} + h\vec{x}) - f(\vec{x})}{h} \\ &= \lim_{h\to 0} \frac{f((1+h)\vec{x}) - f(\vec{x})}{h} \end{aligned}$$

Using homogeneity:

$$\begin{aligned} &= \lim_{h\to 0} \frac{(1+h)^m f(\vec{x}) - f(\vec{x})}{h} \\ &= \left(\lim_{h\to 0} \frac{(1+h)^m - 1}{h}\right) f(x) \end{aligned}$$

Using binomial expansion:

$$\begin{aligned} &= \lim_{h\to 0} \left(\frac{1 + \binom{m}{1} h + \binom{m}{2} h^2 + \ldots + \binom{m}{m} h^m - 1}{h} \right) f(x) \\ &= \lim_{h\to 0} \left(m + \binom{m}{2} h^1 + \ldots + \binom{m}{m} h^{m-1} \right) f(x) \\ &= (m + 0 + \ldots + 0)\, f(x) \\ &= m f(x) \end{aligned}$$

Alternatively, once we have reached $\sum_{i=1}^{n} x_i Df(x) = \left(\lim_{h\to 0} \frac{(1+h)^m - 1}{h}\right) f(x)$, we can define a new function $g(x) = x^m$ and proceed:

$$\begin{aligned}
\sum_{i=1}^{n} x_i Df(x) &= \left(\lim_{h\to 0} \frac{(1+h)^m - 1}{h}\right) f(x) \\
&= \left(\lim_{h\to 0} \frac{(1+h)^m - 1^m}{h}\right) f(x) \\
&= \left(\lim_{h\to 0} \frac{g(1+h) - g(1)}{h}\right) f(x) \\
&= g'(1) f(x) \\
&= m1^{m-1} f(x) \\
&= m f(x)
\end{aligned}$$

□

Problem 2.

Part i.

Let $f : \mathbb{R}^2 \to \mathbb{R}$. We will define $g : \mathbb{R}^2 \to \mathbb{R}^2$ to be the change of variables map:

$$g\begin{pmatrix} r \\ \theta \end{pmatrix} := \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

The composition $f \circ g$ is thus f written in polar coordinates. It is thus reasonable to use the notation that $x := r\cos\theta$, $y := r\sin\theta$, and $r := \sqrt{x^2+y^2}$. We now note that

$$\mathfrak{J}g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{bmatrix}$$

We can now compute:

$$\begin{aligned}
\mathfrak{J}(f\circ g)\begin{pmatrix} r \\ \theta \end{pmatrix} &= \left[\mathfrak{J}f\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}\right] \cdot \left[\mathfrak{J}g\begin{pmatrix} r \\ \theta \end{pmatrix}\right] \\
&= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -y \\ \frac{y}{\sqrt{x^2+y^2}} & x \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f}{\partial x}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial f}{\partial y}\frac{y}{\sqrt{x^2+y^2}} & \frac{\partial f}{\partial y}x - \frac{\partial f}{\partial x}y \end{bmatrix}
\end{aligned}$$

□

Part ii.

Claim 2.1. Let f be a differentiable function $\mathbb{R}^2 \to \mathbb{R}$. Then f can be written as $f(x,y) = \varphi(x^2+y^2)$ for some $\varphi : \mathbb{R} \to \mathbb{R}$ iff $x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = 0$.

$\Rightarrow$. Assume f can be written as $f(x,y) = \varphi(x^2+y^2)$. We can calculate the partial derivatives of f using the chain rule:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2x\varphi'(x^2+y^2) \\
\frac{\partial f}{\partial y} &= 2y\varphi'(x^2+y^2)
\end{aligned}$$

We then see that

$$x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = 2xy\varphi'(x^2+y^2) - 2xy\varphi'(x^2+y^2) = 0$$

as claimed.

$\Leftarrow$. Assume f satisfies $x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = 0$. Then we use our result from the first part of this problem to calculate:

$$\frac{\partial f}{\partial \theta} = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = 0$$

Thus, the value of f does not change with a change in θ. Consequently, it depends only on r which, in turn, depends only on $\left(x^2+y^2\right)$.

□

Problem 3.

Claim 3.1. Let f and g be differentiable maps $\mathbb{R} \to \mathbb{R}^3$. The cross product of f and g is a function $f \times g : \mathbb{R} \to \mathbb{R}^3$ given by $(f \times g)(x) := f(x) \times g(x)$. Then $\mathfrak{J}(f \times g)(x) = \mathfrak{J}f(x) \times g(x) + f(x) \times \mathfrak{J}g(x)$.

For clarity, let's define f and g component-wise:

$$f(x) =: \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} \quad \text{and} \quad g(x) =: \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix}$$

Then, using the definition of the cross product,

$$(f \times g)(x) = \begin{pmatrix} f_2(x)g_3(x) - g_2(x)f_3(x) \\ -f_1(x)g_3(x) + g_1(x)f_3(x) \\ f_1(x)g_2(x) - g_1(x)f_2(x) \end{pmatrix}$$

We can then differentiate using the single-variable product rule (and, of course, the fact that f and g are differentiable):

$$\begin{aligned} \mathfrak{J}(f \times g)(x) &= \begin{pmatrix} f_2(x)g_3'(x) + f_2(x)g_3(x) - g_2(x)f_3'(x) - g_2'(x)f_3(x) \\ -f_1'(x)g_3(x) - f_1'(x)g_3(x) + g_1(x)f_3'(x) + g_1'(x)f_3(x) \\ f_1(x)g_2'(x) + f_1'(x)g_2(x) - g_1(x)f_2'(x) - g_1'(x)f_2(x) \end{pmatrix} \\ &= \begin{pmatrix} f_2'(x)g_3(x) - g_2(x)f_3'(x) \\ -f_1'(x)g_3(x) + g_1(x)f_3'(x) \\ f_1'(x)g_2(x) - g_1(x)f_2'(x) \end{pmatrix} + \begin{pmatrix} f_2(x)g_3'(x) - g_2'(x)f_3(x) \\ -f_1(x)g_3'(x) + g_1'(x)f_3(x) \\ f_1(x)g_2'(x) - g_1'(x)f_2(x) \end{pmatrix} \\ &= \mathfrak{J}f(x) \times g(x) + f(x) \times \mathfrak{J}g(x) \end{aligned}$$

□