Problem Set 5, Part C Solution Set

Tony Várilly

Math 25a, Fall 2001

1. (a) Prove that a locally Lipschitz function is continuous.

Solution. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipshictz function. We know this means for every $x \in \mathbb{R}^n$ there exist positive M and δ such that

$$|f(x) - f(y)| < M|x - y| \quad \text{whenever } |x - y| < \delta.$$

Now let $\epsilon > 0$ be given, and let $\delta' = \min\{\delta, \epsilon/M\}$. If $\delta' = \epsilon/M$, then $|x - y| < \epsilon/M < \delta$, so that $M|x - y| < \epsilon$, and by the Lipschitz condition, we get

$$|f(x) - f(y)| < M|x - y| < \epsilon \quad \text{whenever } |x - y| < \delta',$$

which means f is continuous. If, on the other hand, $\delta' = \delta$, then we have $|x - y| < \delta < \epsilon/M$, and so $M|x - y| < \epsilon$ again, and continuity follows like before. □

Remark. Many people had an argument similar to the above, but simply said "let $\delta = \epsilon/M$." This need not always work, since δ and M might depend on each other, and we have to be *free to choose any* $\epsilon > 0$.

(b) Prove that a differentiable function is locally Lipschitz.

Solution (due to John Provine). If f is differentiable, then for each $x \in \mathbb{R}^n$ there exists a linear function $Df(x)$ such that

$$\lim_{h \to 0} \frac{1}{|h|}(f(x + h) - f(x) - Df(x)(h)) = 0.$$

Substituting $h \equiv x - y$ gives

$$\lim_{y \to x} \frac{1}{|x - y|}(f(x) - f(y) - Df(x)(x - y)) = 0.$$

Then, from the definition of the limit, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y) - Df(x)(x - y)| < \epsilon|x - y| \quad \text{whenever } |x - y| < \delta.$$

In particular, this holds for $\epsilon = 1$. From the triangle inequality, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \leq |f(x) - f(y) - Df(x)(x - y)| + |Df(x)(x - y)| < |x - y| + |Df(x)(x - y)|$$

whenever $|x-y|<\delta$. From problem set #4, since $Df(x)$ is a linear map, there exists a constant M_0 such that

$$|Df(x)(x-y)| \leq M_0|x-y|.$$

Setting $M = 1 + M_0$, it follows that

$$|f(x)-f(y)| < |x-y| + M_0|x-y| = (1+M_0)|x-y| = M|x-y|$$

whenever $|x-y|<\delta$, which means that f is locally Lipschitz. □

(c) Show that the two implications above are strict.

Solution. First, we will give an example of a function that is continuous but not locally Lipschitz. The most popular example was $f:\mathbb{R}\to\mathbb{R}$, such that $f(x) = x^{1/3}$. This is a continuous function (it is *not* a polynomial—polynomials have non-negative integer exponents). We'll show that f is not locally Lipschitz at $x=0$.

Suppose f is locally Lipschitz at this point. Let M and δ be the two constants such that $|y^{1/3}| < M|y|$, for $|y|<\delta$. We'll show one can always find a y_0 such that $|y_0^{1/3}| \geq M|y_0|$ and $|y_0|<\delta$. We will find a positive y_0, so we can drop the absolute value signs. The condition $y^{1/3} \geq My$ is equivalent to $1/M^{3/2} \geq y$. Let $y_0 = \min\{1/M^{3/2}, \delta/2\}$. This will also guarantee that $y_0 < \delta$. So f is not locally Lipschitz at $x=0$.

Now we'll show an example of a function that is locally Lipschitz but not differentiable. The easiest example is $f:\mathbb{R}\to\mathbb{R}$ such that $f(x) = |x|$. We know from one-variable calculus that f is not differentiable at $x=0$. Indeed, recall that

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = -1, \quad \text{yet} \quad \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = 1.$$

Our function, however, is locally Lipschitz. If $x<0$ or $x>0$ then the slope of the function has absolute value 1 in a small enough region. That is $|f(x)-f(y)| = |x-y| < \delta$ for a small enough δ (you can work out the details if you want). So any $M>1$ will do for this δ. If $x=0$, we have $|f(y)| = ||y|| = |y|$. Any $M>1$ and any $\delta>0$ will do the trick for local 'Lipschitzticity'.

□

2. Show that the generalization of the mean value theorem for functions $f:\mathbb{R}^n\to\mathbb{R}^m$ is not true. Is it true that for any differentiable map $f:\mathbb{R}\to\mathbb{R}^2$ and any two points $a,b\in\mathbb{R}$ there exists $c\in[a,b]$ and $\lambda\in\mathbb{R}$ such that

$$f(b)-f(a) = \lambda Df(c)(b-a)?$$

Solution. We provide a counterexample for both claims. Consider the map $f:\mathbb{R}\to\mathbb{R}^2$ that maps $t \mapsto \begin{pmatrix} t^3 \\ t^6 \end{pmatrix}$. The partial derivatives $Df_1(t)$ and $Df_2(t)$ exist and are continuous (they are polynomials), so f is differentiable and

$$Df(t) = \begin{bmatrix} 3t^2 \\ 6t^5 \end{bmatrix}.$$

Suppose the mean value theorem holds. Let $a = 0, b = 1$. Then there is a $c \in [0,1]$ for which

$$\begin{aligned} f(1) - f(0) &= \begin{bmatrix} 3c^2 \\ 6c^5 \end{bmatrix} (1 - 0), \\ \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 3c^2 \\ 6c^5 \end{bmatrix}. \end{aligned}$$

However, no $c \in [0,1]$ can satisfy the required conditions simultaneously.

The same example works for the second part of the question. This time, however, use the interval $[-k, k], k \neq 0$ instead of $[0,1]$. We are required to find a c and a λ such that

$$\begin{aligned} f(k) - f(-k) &= \begin{bmatrix} 3c^2 \\ 6c^5 \end{bmatrix} (k - -k), \\ \Rightarrow \begin{bmatrix} 2k^3 \\ 0 \end{bmatrix} &= \lambda \cdot \begin{bmatrix} 3c^2 \\ 6c^5 \end{bmatrix} \cdot 2k. \end{aligned}$$

The second component of the vector tells us that $0 = 12\lambda c^5 k$. Since $k \neq 0$, we must have either $c = 0$ or $\lambda = 0$. But then $0 = 6\lambda c^2 k = 2k^3$ contradicting the assumption that $k \neq 0$. $\square$