MATH 25A – PROBLEM SET #6
DUE FRIDAY NOVEMBER 2

1. Part A

(1) Problem 2.2.8 in the textbook.
(2) Problem 2.3.2 in the textbook.
(3) Problem 2.1.3 in the textbook.

2. Part B

For this part of the homework, please read about elementary matrices on pgs 163-165. The book shows that an elementary row operation of A can be given as a multiplication EA of A with an elementary matrix E. One can also define elementary column operations, which are given as multiplications AE of A with E from the right.

(1) A matrix $A \in Mat(n,n)$ is called *upper triangular* if $A_{ij} = 0$ for $i > j$, and *lower triangular* if $A_{ij} = 0$ for $i < j$. Prove that if A and B are upper triangular (resp. lower triangular), then AB is upper triangular (resp. lower triangular).
(2) A matrix $P \in Mat(n,n)$ is called a *permutation matrix* if the entries of P are zeroes and ones, and every row and every column of P contains exactly one one. Multiplying a matrix A with a permutation matrix from the left has the effect of permuting the rows of A. Prove that if A is any matrix then we can write

$$PA = LU$$

for some permutation matrix P, lower triangular matrix L, and upper triangular matrix U. (Hint: consider the row reduction in matrix form. First do the case where you don't have to exchange any rows.)
(3) Prove that for any matrix A there exist invertible matrices P and Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

for some r. (Hint: use row and column reduction.)

3. Part C

Recall that if f is a continuous map, then inverse images of closed sets are closed, inverse images of open sets are open, and images of compact sets are compact. A continuous map f is called:

- *open* if $f(U)$ is open for all open sets U;
- *closed* if $f(C)$ is closed for all closed sets C;
- *compact* if $f^{-1}(K)$ is compact for all compact sets K.

(1) Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection to the first factor, $\pi(x,y) = x$. Prove that π is open, but not closed and not compact. (Note that if a closed set is bounded, then its image is automatically closed. Hence, to prove that π is not closed, look for an unbounded closed set.)
(2) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^3 - x$. Prove that f is closed and compact, but not open. (Hint: to show that $f(C)$ is closed for any closed C, write $\mathbb{R} = C_1 \cup C_2$ for some closed sets C_1, C_2, such that both $f(C \cap C_1)$ and $f(C \cap C_2)$ are closed.)

(3) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a closed map, such that all fibers $f^{-1}(y)$ for $y \in \mathbb{R}^m$ are compact.

(a) Prove that if $f^{-1}(y) \subset U$ for some open $U \subset \mathbb{R}^n$, then there exists an ε-ball $B_\varepsilon(y)$ such that $f^{-1}(B_\varepsilon(y)) \subset U$.

(b) prove that f is compact. (Hint: use the open cover property of compact sets.)