# Problem Set 6, Part A Solution Set

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1. Problem 2.2.8: Given the system of equations

$$\begin{aligned} x_1 - x_2 - x_3 - 3x_4 + x_5 &= 1 \\ x_1 + x_2 - 5x_3 - x_4 + 7x_5 &= 2 \\ -x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 &= 0 \\ -2x_1 + 5x_2 - 4x_3 + 9x_4 + 7x_5 &= \beta, \end{aligned}$$

for what values of $\beta$ does the system have a solution? When solutions exist, give values of pivotal variables in term of the non-pivotal variables.

*Solution.* Solving this system of equations amounts to row-reducing the augmented matrix

$$\begin{bmatrix} 1 & -1 & -1 & -3 & 1 & 1 \\ 1 & 1 & -5 & -1 & 7 & 2 \\ -1 & 2 & 2 & 2 & 1 & 0 \\ -2 & 5 & -4 & 9 & 7 & \beta \end{bmatrix}.$$

First, we use row operations to clear the first column:

$$\begin{bmatrix} 1 & -1 & -1 & -3 & 1 & 1 \\ 0 & 2 & -4 & 2 & 6 & 1 \\ 0 & 1 & 1 & -1 & 2 & 1 \\ 0 & 3 & -6 & 3 & 9 & \beta+2 \end{bmatrix}.$$

Now we divide the second row by 2 and use it to clear up the second column of the matrix:

$$\begin{bmatrix} 1 & 0 & -3 & -2 & 4 & 3/2 \\ 0 & 1 & -2 & 1 & 3 & 1/2 \\ 0 & 0 & 1 & -2/3 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & \beta+1/2 \end{bmatrix}.$$

Finally, we clean up the third column to get

$$\begin{bmatrix} 1 & 0 & 0 & -4 & 3 & 2 \\ 0 & 1 & 0 & -1/3 & 7/3 & 5/6 \\ 0 & 0 & 1 & -2/3 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & \beta+1/2 \end{bmatrix}.$$

By a theorem we proved in class, the original set of equations will have a solution if and only if the row-reduced echelon form of the system's augmented matrix has no pivotal 1 in the last column. So the system will only have a solution when $\beta = -1/2$. In this case, we see that $x_1$, $x_2$ and $x_3$ will be our pivotal variables, and

$$\begin{aligned} x_1 &= 2 + 4x_4 - 3x_5 \\ x_2 &= 5/6 + 1/3\, x_4 - 7/3\, x_5 \\ x_3 &= 1/6 + 2/3\, x_4 + 1/3\, x_5. \end{aligned}$$

□

**Remark.** Many people avoided writing out their computations. This is normally ok, but it is not when the whole point of the exercise is to give you a feel for what a computation is like.

2. Problem 2.3.2

(a) Row reduce symbolically the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & a \\ 1 & -1 & 1 & b \\ 1 & 1 & 2 & c \end{bmatrix}.$$

*Solution.* We start by dividing the first row by 2 and then clearing the first column of A:

$$\begin{bmatrix} 1 & 1/2 & 3/2 & a/2 \\ 0 & -3/2 & -1/2 & -a/2 + b \\ 0 & 1/2 & 1/2 & -a/2 + c \end{bmatrix}.$$

Next, we divide the second row by $-3/2$ and use this row to clear the second column:

$$\begin{bmatrix} 1 & 0 & 4/3 & a/3 + b/3 \\ 0 & 1 & 1/3 & a/3 - 2b/3 \\ 0 & 0 & 1/3 & -2a/3 + b/3 + c \end{bmatrix}.$$

Finally, we divide the thrid row by $1/3$ and use this row to clear the third column. We get the desired row reduced form:

$$\begin{bmatrix} 1 & 0 & 0 & 3a - b - 4c \\ 0 & 1 & 0 & a - b - c \\ 0 & 0 & 1 & -2a + b + 3c \end{bmatrix}.$$

□

(b) Compute the inverse of the matrix

$$B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

*Solution.* The approach we take is by row-reducing the matrix $[B \mid I_3]$. We'll show in our steps the matrix obtained by clearing one column at a time.

$$\begin{bmatrix} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & -3/2 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 1/2 & -1/2 & 0 & 1 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & 4/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 1/3 & 1/3 & -2/3 & 0 \\ 0 & 0 & 1/3 & -2/3 & 1/3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & -1 & -4 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{bmatrix}$$

Thus,

$$B^{-1} = \begin{bmatrix} 3 & -1 & -4 \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$$

□

(c) What is the relation between the answers in parts (a) and (b)?

*Solution.* The idea here is to notice that if $B$ is an invertible matrix and $v$ any vector of the same dimension, Then $[B \mid v]$ row reduces to $[I \mid B^{-1}v]$. □

3. Problem 2.1.3 Show that the row operaion that consists of exchanging two rows is not necessary.

*Solution.* Consider the matrix

$$\begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix},$$

where each $v_i$ is a row vector of the matrix. Suppose without loss of generality that $i > j$ and that we wnat to exchange the $i^{\text{th}}$ and $j^{\text{th}}$ rows. We can do so using only the other two kinds of row operations:

$$\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_i & - \\ & \vdots & \\ - & v_j & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} \to \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_i & - \\ & \vdots & \\ - & v_i + v_j & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} \to \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & -v_j & - \\ & \vdots & \\ - & v_i + v_j & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} \to \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & -v_j & - \\ & \vdots & \\ - & v_i & - \\ & \vdots & \\ - & v_n & - \end{bmatrix} \to \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_j & - \\ & \vdots & \\ - & v_i & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

□