Math 25a Solution Set #6 (Part B)

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Problem 1

We show here that the product of two upper triangular matrices is again upper triangular. The proof for lower triangular matrices is essentially the same. Let $A, B \in Mat(n, n)$ be upper triangular, and let $[A]_{ij} = a_{ij}$, $[B]_{ij} = b_{ij}$, so that for $i > j$ $a_{ij} = b_{ij} = 0$. We want to show AB is upper triangular too, so all we need to do is check that for $i > j$ $[AB]_{ij} = 0$ as well. But we have:

$$[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Each summand in the sum above has to be 0, since we have:

$$\begin{aligned} a_{ik}b_{kj} \neq 0 \quad &\Rightarrow \quad a_{ik} \neq 0 \text{ and } b_{kj} \neq 0 \\ &\Rightarrow \quad i \leq k \text{ and } k \leq j \Rightarrow i \leq j \end{aligned}$$

But we know $i > j$, so $a_{ik}b_{kj} \neq 0$ cannot happen. Thus, $[AB]_{ij} = 0$ whenever $i > j$, and AB is upper triangular.

Problem 2

Recall the definition of a permutation matrix from the problem set, and the fact that permuting the rows of a matrix can be achieved by multiplying it from the right by a permutation matrix.

We show here that for any matrix A we can write $PA = LU$ for some permutation matrix P, lower triangular matrix L, and upper triangular matrix U. The idea is to permute rows of A so as to get a convinient matrix PA which can be row reduced to an upper triangular matrix by using only lower triangular elementary matrices. Once you show this can be done, the textbook tells you that the inverse of a lower triangular elementary matrix is still lower triangular, the product of any number of lower triangular matrices is still lower triangular (by the previous problem), and it follows that:

$$(E_1 \ldots E_k)(PA) = U \Rightarrow PA = E_k^{-1} \ldots E_1^{-1} U = LU$$

provided all E_i's are lower triangular.

The rest of the problem is just devising a procedure that will do the job using just lower triangular matrices. Note that, except row switches, which are neither lower nor upper triangular, the only elementary matrix which is not lower triangular is the one that corresponds to adding a multiple of a row to a row above it. So the idea is to devise a way to avoid such moves. Consider the following procedure:

(1) Set $j = 1$.

(2) Find the smallest $i \geq j$ such that i-th row has a non-zero element in the j-th column. If no such i exists, go on to step (6), else continue.

(3) Switch the i-th and the j-th row.

(4) Add multiples of the new j-th row (previously i-th) to all other rows **below** it which have a nonzero entry in the j-th column, so as to get zero entries in the part of the j-th column of A below the main diagonal.

(5) Divide the new j-th row so that the (j, j) entry is 1.

(6) Repeat for $j + 1$.

Step(4) guarantees that we end up with an upper triangular matrix. Now, if we are dealing with a particularly nice matrix A, i.e. such that $i = j$ at every step, we need no switches to carry out the above procedure, and we get a row reduction of A using only lower triangular matrices, and hence we can let $P = I$ and the statement of the problem follows as discussed above.

But the crucial thing to notice here was that switches can be performed at the beginning. For if we apply the above procedure - minus step (4) - to any matrix, what we are doing is just switching arround the rows of A, and hence in effect only multiplying A by some permutation matrix P. But If we consider the matrix PA, we notice this matrix is nice – if we were to apply the entire procedure to it, we would need no switches, since we have rearranged the rows of A into the order they are in in PA precisely in sucha way that we have $i = j$ (if i exists) in PA at each step. Thus we conclude that in row reduction of PA no switches would be used, and hence $PA = LU$ by the discussion at the beginning of the problem.

Problem 3

We now show that for any matrix A there exist invertible matrices P and Q such that

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

for some r.

The hint tells you to use column and row reduction. Given the "nastiness" of the previous problem, it makes sense to use it here. The previous problem gives us the existence of invertible matrices L^{-1}, P such that $L^{-1}PA = U$ is upper triangular, with ones and zeros at the diagonal. It is left to show the existence of an invertible matrix Q such that:

$$UQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Since column reduction is just the multiplication by elementary matrices on the right, we will column reduce U, but we need to do it so that it remains upper triangular, if we want to ensure the end-product is to be what we need. Thus, since adding a multiple of a column to a column to its left may create some nonzero entries below the main diagonal, some care is needed here. Our job is easier here than in the previous problem, for U is already upper-triangular. The following procedure does the job:

(1) set $j = 1$.

(2) if the j-th column is zero, go to (3). Else add multiples of it to the columns **to the right of it** so as to get zero entries in the part of the j-th row to the right of the main diagonal.

(3) repeat for $j + 1$.

Note that the matrix stays upper triangular throughout, but we end up with a diagonal matrix with only ones on the diagonal, which is what we wanted.