Problem Set #6c – Solutions

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Problem 1.

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection to the first factor (i.e., $\pi((x, y)) = x$).

Claim 1.1. π is an open map.

Let $U \subset \mathbb{R}^2$ be open. Let $x \in \pi(U)$. I claim that there exists an open ball around x contained in $\pi(U)$. Becuase $x \in \pi(U)$, there exists some $y \in U$ such that $\pi(y) = x$. Then, because U is open, there exists an open ball $B_\epsilon(x) \subset U$ (where $B_\epsilon := \{p \in \mathbb{R}^2 \mid ||x - p|| < \epsilon\}$). Now, note that $\pi(B_\epsilon(x)) = B_\epsilon(y)$ (where $B_\epsilon(y) := \{p \in \mathbb{R} \mid ||y - p|| < \epsilon\}$), and thus, $y \in B_\epsilon(y) \subset \pi(U)$. Thus $\pi(U)$ is open.

□

Claim 1.2. π is not a closed map.

Consider $U := \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$. Clearly, U is closed: every point in its complement ($\mathbb{R}^2 \setminus U$) has a finite distance from the set U, and thus an open ball with radius that distance is contained completely in the complement of U. However, $\pi(U) = \mathbb{R} \setminus \{0\}$ (0 being the only element of $\mathbb{R}$ without a multiplicative inverse) which is not closed, as its complement, $\{0\}$ is not open as it contains no open neighborhood of the point 0. Thus U is a closed set whose image is not closed, thus π is not a closed map.

□

Claim 1.3. π is not a compact map.

Consider the set $C := \{0\} \subset \mathbb{R}$. Being a single point, C is closed and bounded and therefore compact. However, $\pi^{-1}(C) = \{(0, x) \mid x \in \mathbb{R}\}$, which is not bounded and therefore not compact. Thus C is a compact set whose preimage under π is not compact, thus π is not a compact map.

□

Problem 2.

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f : x \mapsto x^3 - x$.

Claim 2.1. f is a closed map.

We begin by noting that f has a local maximum at $-\frac{\sqrt{3}}{3}$ and a local minimum at $frac\sqrt{3}3$ (as can be verified by finding the roots of $\frac{df}{dx}$). Using this fact, we will create three closed intervals:

$$\begin{aligned} C_1 &:= \left(-\infty, -\frac{\sqrt{3}}{3}\right] \\ C_2 &:= \left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right] \\ C_3 &:= \left[\frac{\sqrt{3}}{3}, \infty\right) \end{aligned}$$

We note that $C_1 \cup C_2 \cup C_3 = \mathbb{R}$. Now, let C be a closed subset of $\mathbb{R}$. Then

$$\begin{aligned} f(C) &= f(\mathbb{R} \cap C) \\ &= f((C_1 \cup C_2 \cup C_3) \cap C \\ &= f(C_1 \cap C) \cup f(C_2 \cap C) \cup f(C_3 \cap C) \end{aligned}$$

We will prove that $f(C)$ is closed by proving that $f(C_i \cap C)$ is closed (for $i \in \{1, 2, 3\}$), for then $f(C)$ would (by the above) be a finite union of closed sets and therefore closed.

1. $f(C_1 \cap C)$ **and** $f(C_3 \cap C)$ **are closed**. We define a new map $tilf1 : C_1 \to \left(-\infty, f\left(\frac{\sqrt{3}}{3}\right)\right]$ by $g_1(x) = f(x)$. It is clear that $g_1(C_1 \cap C) = f(C_1 \cap C)$. By examining the graph of g_1 (e.g. applying the "horizontal line test" and exchanging the axes to obtain a graph of g_1^{-1}), we observe that g_1 is not only bijective and continuous, but has a continuous inverse. *(For those of you who have studied some additional topology, g_1 is a homeomorphism.)* The fact that g_1 is bijective implies that $g_1^{-1}(g_1(C_1 \cap C)) = C_1 \cap C$, and therefore the continuity of g_1^{-1} implies that $g_1(C_1 \cap C)$ is closed, as it is the pre-image of the closed set $C_1 \cap C$ under the continuous map g_1^{-1}. Thus $f(C_1 \cap C)$ is also closed. Because f is an odd function, an almost-identical method works for proving that $f(C_3 \cap C)$ is closed.

2. $f(C_2 \cap C)$ **is closed**. We note that C_2 is closed and bounded and therefore compact. Thus $C_2 \cap C$ is a closed subset of a compact set and therefore compact itself. Because f is continuous, the image of a compact set is compact, and therefore $f(C_2 \cap C)$ is compact and thus closed.

□

Claim 2.2. f is a compact map.

Let C be a compact subset of $\mathbb{R}$. We must show that $f^{-1}(C)$ is compact. We will do this by proving that $f^{-1}(C)$ is closed and bounded.

1. **Closed.** Because it is compact, C is closed, and because it is a polynomial, f is continuous. The pre-image of a closed set under a continuous map is closed, thus $f^{-1}(C)$ is closed.

2. **Bounded.** Because it is compact, C is bounded, so we can select a some value b such that $|x| < b \forall x \in C$. To make life easier, however, let's select a b that is greater than 2. Because our b is greater than or equal to 2, we see that whenever $|x| \geq b$,

$$|f(x)| > |x| \geq b$$

Thus, b is a bound on $f^{-1}(C)$: if, to the contrary, there were some $y \in f^{-1}(C)$ such that $|y| \geq b$, then we would have $|f(y)| \geq b$ and therefore $f(y) \notin C$ which would contradict $y \in f^{-1}(C)$.

□

Claim 2.3. f is not an open map.

Let U be the open interval $(-1, 1)$. But, $f(U) = \left[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right]$, which is clearly closed. Thus, U is an open set whose image under f is not open, thus f is not an open map.

□

Problem 3.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a closed map such that all fibers $f^{-1}(y)$ are compact (for all $y \in \mathbb{R}^m$).

Part i.

Claim 3.1. Let $U \subset \mathbb{R}^n$ be open and let $y \in \mathbb{R}^m$ such that $f^{-1}(y) \subset U$. Then $\exists \epsilon > 0$ such that $f^{-1}(B_\epsilon(y)) \subset U$.

Because U is open, its complement, $\mathbb{R}^n \setminus U$ is closed, and therefore because f is a closed map $f(\mathbb{R}^n \setminus U)$ is also closed. Thus its complement, $V := \mathbb{R}^m \setminus f(\mathbb{R}^n \setminus U)$ is open. Intuitively, $f(\mathbb{R}^n \setminus U)$ is the image of all points in U, and thus V is the set of all points whose pre-image is a subset of U. Because V is open, it must contain an open ball around y.

Formally, y must be an element of V, for its pre-image is a subset of U and therefore cannot intersect the complement of U. Thus $y \in V$ so $\exists \epsilon > 0$ s.t. $B_\epsilon(y) \subset V$. Now, assume contrary to the claim that $f^{-1}(B_\epsilon(y))$ is not a subset of U. Then, there must exist some $x \in (\mathbb{R}^n \setminus U) \cap f^{-1}(B_\epsilon(y))$. But then $f(x) \in f(\mathbb{R}^n \setminus U)$, so $f(x) \notin V$. Because $B_\epsilon(y) \subset V$, $f(x) \notin B_\epsilon(y)$, meaning $x \notin f^{-1}(B_\epsilon(y))$, which is a contradiction.

□

Part ii.

Claim 3.2. f is a compact map.

Let $C \subset \mathbb{R}^m$ be compact. Let $\mathfrak{U}$ be an open cover of $f^{-1}(C)$. We must prove that $\mathfrak{U}$ contains a finite subcover, $\mathfrak{U}'$ of $f^{-1}(C)$.

For any $y \in C$, $f^{-1}(y) \subset f^{-1}(C)$, and thus $\mathfrak{U}$ is also an open cover of $f^{-1}(y)$. Now, by the hypothesis of the problem, $f^{-1}(y)$ is compact, so $\mathfrak{U}$ contains a finite subcover of $f^{-1}(y)$, call it $\mathfrak{U}_y$. The union of this finite subcover, $\cup\mathfrak{U}_y$, must be open (it is the union of open sets), and thus by the first part of this problem, $\exists \epsilon_y > 0$ such that $f^{-1}(y) \subset (\cup\mathfrak{U}_y)$.

Now, the set of all such open balls, $\mathfrak{B} := \{B_{\epsilon_y}(y) \mid y \in C\}$ is clearly an open cover of C, thus it contains a finite subcover, $\mathfrak{B}'$. Finally, we can construct our finite subcover of $\mathfrak{U}$: Define

$$\mathfrak{U}' := \cup_{\{y \in C \mid B_{\epsilon_y}(y) \in \mathfrak{B}'\}} \mathfrak{U}_y$$

that is, for each $y \in C$ whose neighborhood $B_{\epsilon_y}(y)$ was included in $\mathfrak{B}'$, add to $frakU'$ the finite subcover $\mathfrak{U}_y$ of $f^{-1}(y)$. As the finite union of finite sets, $\mathfrak{U}'$ must be finite. Moveover, it covers $f^{-1}(C)$ because it covers $f^{-1}(B_{\epsilon_y})$ for all $B_{\epsilon_y} \in \mathfrak{B}'$, and $\mathfrak{B}'$ covers C.

□