MATH 25A – PROBLEM SET #7
DUE FRIDAY NOVEMBER 9

1. Part A

(1) Find a basis for $Mat(n, m)$. If V is a vector space with basis $\{v_1, \ldots, v_n\}$ and W is a vector space with basis $\{w_1, \ldots, w_n\}$, find a basis for the space $Lin(V, W)$.
(2) Problem 2.4.12(a) in the textbook.
(3) Prove that $\{v_1, \ldots, v_n\}$ is a basis of V if and only if for any vector space W and any vectors $w_1, \ldots, w_n \in W$ there exists a unique linear map $L : V \to W$ such that $L(v_i) = w_i$ for $i = 1, \ldots, n$.
(4) Let V and W be subspaces of $\mathbb{R}^n$. Define $V + W = \{v + w | v \in V, w \in W\}$. It is easy to see that this is a subspace of $\mathbb{R}^n$. Prove that

$$\dim V + W = \dim V + \dim W - \dim V \cap W.$$

2. Part B

(1) Problem 2.5.15 in the textbook.
(2) Problem 2.5.16 in the textbook.

3. Part C

(1) Let V be a vector space with basis $\{v_1, \ldots, v_n\}$ and W a vector space with basis $\{w_1, \ldots, w_m\}$. Also let $L : V \to W$ be a linear map with matrix A with respect to the two bases.
  (a) Prove that $\{w_1, \ldots, w_{i-1}, \frac{1}{\alpha} w_i, w_{i+1}, \ldots, w_m\}$ for any $1 \leq i \leq m$ and $\alpha \neq 0$ is again a basis of W. Find the matrix of L with respect to this new basis in terms of A.
  (b) Prove that $\{w_1, \ldots, w_{i-1}, w_i - \beta w_j, w_{i+1}, \ldots, w_m\}$ for any $1 \leq i \neq j \leq m$ and $\beta \in \mathbb{R}$ is again a basis of W. Find the matrix of L with respect to this new basis in terms of A.
  (c) Let A' be a matrix obtained from A by row reduction. Prove that A' is the matrix of L with respect to the bases $\{v_1, \ldots, v_n\}$ of V and $\{w'_1, \ldots, w'_m\}$ of W. Explain how $\{w'_1, \ldots, w'_m\}$ is obtained from $\{w_1, \ldots, w_m\}$.
  (d) We construct a basis $\{w'_1, \ldots, w'_m\}$ as follows. Let $i_1 > 0$ be the smallest index such that $L(v_{i_1}) \neq 0$ and define $w'_1 = L(v_{i_1})$. Let $i_2 > i_1$ be the smallest index such that $L(v_{i_2}) \notin Span\{w'_1\}$ and define $w'_2 = L(v_{i_2})$. Inductively, let $i_k > i_{k-1}$ be the smallest index such that $L(v_{i_k}) \notin Span\{w'_1, \ldots, w'_{k-1}\}$ and define $w'_k = L(v_{i_k})$. This way we get linearly independent $\{w'_1, \ldots, w'_p\}$, which can be extended to a basis $\{w'_1, \ldots, w'_p, \ldots, w'_m\}$ of W. Prove that the matrix of L with respect to $\{v_1, \ldots, v_n\}$ and $\{w'_1, \ldots, w'_m\}$ is in echelon form.
  (e) Prove that if for any basis $\{w'_1, \ldots, w'_m\}$ of W, the matrix of L with respect to $\{v_1, \ldots, v_n\}$ and $\{w'_1, \ldots, w'_m\}$ is in echelon form, then $\{w'_1, \ldots, w'_m\}$ is constructed as in the previous problem. In particular, $\{w'_1, \ldots, w'_p\}$ are uniquely determined by L and $\{v_1, \ldots, v_n\}$.
  (f) Prove that the row reduction of A to a matrix A' in echelon form is unique.

(2) Consider a sequence of linear maps between vector spaces:

$$0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \dots \xrightarrow{d_{k-1}} V_k \xrightarrow{d_k} 0$$

This sequence is called *exact* if $Ker(d_i) = Img(d_{i-1})$ for all $1 \leq i \leq k$. (Note that the zeroes at both ends are the zero vector spaces, and the maps d_0 and d_k are the obvious ones. Then the condition $Ker(d_1) = Img(d_0) = 0$ means that d_1 is injective, and $Img(d_{k-1}) = Ker(d_k) = V_k$ means that d_{k-1} is surjective.)

(a) Consider the short exact sequence ($k = 3$):

$$0 \rightarrow V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} V_3 \rightarrow 0$$

Prove that $\dim V_2 = \dim V_1 + \dim V_3$. (Hint: identify the image and the kernel of d_2.)

(b) More generally, prove that the Euler characteristic of an exact sequence is zero:

$$\sum_i (-1)^i \dim V_i = 0$$

(Hint: prove that the following two sequences are exact:

$$0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \dots \xrightarrow{d_{k-2}} Img(d_{k-2}) \rightarrow 0$$

$$0 \rightarrow Img(d_{k-2}) \rightarrow V_{k-1} \rightarrow V_k \rightarrow 0$$

and use induction on k.)