Problem Set #7a – Solutions

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Problem 1.

We will first prove a lemma:

Claim 1.1. Let P and Q be vector spaces, let $\varphi : P \to Q$ be a (vector space) isomorphism (i.e. a bijective linear map), and let $\mathfrak{P} := \{p_i \mid i \in I\}$ be a basis of P. Then the image of $\mathfrak{P}$ under φ, $\{\varphi(p_i) \mid i \in I\}$ is a basis of Q.

We must prove that $\{\varphi(p_i) \mid 1 \leq i \leq n\}$ is *linearly independent* and *spans*.

Linear Independence. Suppose, contrary to the claim, that

$$\sum_{i \in I} \alpha_i \varphi(p_i) = 0$$

where a positive, finite number of α_i are non-zero. Then by linearity,

$$0 = \sum_{i \in I} \alpha_i \varphi(p_i) = \varphi\left(\sum_{i \in I} p_i\right)$$

But the pre-image of 0 under an isomorphism is simply 0, thus $\sum_{i \in I} p_i = 0$ which contradicts the hypothesis that $\mathfrak{P}$ is linearly-independent.

Spanning. Let $q \in Q$. Then because φ is surjective, $\exists x \in P$ s.t. $\varphi(x) = q$. Because $x \in P$, it can be written as a finite linear combination of elements of $\mathfrak{P}$:

$$p = \sum_{i \in I} \alpha_i p_i$$

(where only finitely many α_i are non-zero). Then:

$$\begin{aligned} q &= \varphi(x) \\ &= \varphi\left(\sum_{i \in I} \alpha_i p_i\right) \\ &= \sum_{i \in I} \alpha_i \varphi(p_i) \end{aligned}$$

(where only finitely many of the α_i are ever nonzero). Thus, $q \in \text{Span}(\varphi(\mathfrak{P}))$.

□

Claim 1.2. Let $\mathfrak{B}_{n,m}$ be the set of all $n \times m$ matrices in which exactly one component is 1 and all other components are 0. Then $\mathfrak{B}_{n,m}$ is a basis for $n \times m$ matrices.

We recall that there is a natural isomorphism between $\text{Mat}(n, m)$ and $\mathbb{R}^{nm}$. We know that the set $\{\vec{e_i} \mid 0 < i < nm\}$ is a basis of $\mathbb{R}^{nm}$, and we observe that the image of this basis under this natural isomorphism is simply $\mathfrak{B}_{n,m}$. Thus, by the lemma proven above, $\mathfrak{B}_{n,m}$ is a basis of $\text{Mat}(n, m)$.

□

Claim 1.3. Let $m_{i,j}$ ($0 < i \leq n$ and $0 < j \leq m$) be the unique linear map given by

$$m_{i,j}(v_k) := \delta_{i,k} w_j$$

(where $\delta_{i,k}$ is the Kronecker delta symbol, defined by $\delta_{a,b} := 1$ if $a = b$ and $\delta_{a,b} := 0$ otherwise.) In other words, $m_{i,j}$ maps v_i to w_j and maps all $v_{l \neq i}$ to zero. Then the set $\{b_{i,j} \mid 0 < i \leq n \text{ and } 0 < j \leq m\}$ is a basis of $\mathrm{Lin}(V, W)$.

We note that the map $\varphi : \mathrm{Lin}(U, V) \to \mathrm{Mat}(n, m)$ which assigns to each linear map its matrix (under the bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$) is a (vector space) isomorphism: it is linear and bijective. We note further that the image of $\mathfrak{B}_{n,m}$ under the inverse of this isomorphism is $\{b_{i,j} \mid 0 < i \leq n \text{ and } 0 < j \leq m\}$, thus $\{b_{i,j} \mid 0 < i \leq n \text{ and } 0 < j \leq m\}$ is a basis of $\mathrm{Lin}(U, V)$.

□

Problem 2.

Claim 2.1. Let V be a finite dimensional vector space, and let $\{v_1, \ldots, v_k\}$ be a linearly independent subset. Then there exist $v_{k+1}, \ldots, v_n$ such that $\{v_1, \ldots, v_n\}$ (for some $n \geq k$) is a basis of V.

Because V is finite dimensional, it contains a finite spanning set, $U := \{u_1, \ldots, u_m\}$. We now provide a method for constructing $v_{k+1}, \ldots, v_n$:

If $u_1 \in \mathrm{Span}(\{v_1, \ldots, v_k\})$, then we let $u_i =: v_{k+1}$. We can now check if $u_2 \in \mathrm{Span}(\{v_1, \ldots, v_k\})$ (or $\mathrm{Span}(\{v_1, \ldots, v_{k+1}\})$ if we have defined a v_{k+1}), and if not, we define $u_2 =: v_{k+2}$. We continue this process, checking to see whether each successive element of U is in the span of our growing list of v_i. Once we have checked them all, I claim that the set $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_{n \geq k}\}$ is a basis: it must span V because its spans contains U, and it must be linearly independent because no v_i is in the span of $\{v_1, \ldots, v_{i-1}\}$.

□

Problem 3.

Claim 3.1. Let V and W be nontrivial vector spaces and let $\{v_1, \ldots, v_n\} \subset V$. Then $\{v_1, \ldots, v_n\}$ is a basis iff for all $(w_1, \ldots, w_n) \in W^n$, there exists a unique linear map, φ, with $\forall i, w_i = \varphi(v_i)$.

$\Rightarrow$**. (Assume $\{v_1, \ldots, v_n\}$ is a basis in order to prove the existence and uniqueness of φ.)** Let $t_1, \ldots, t_m$ be a basis of $\mathrm{Span}(\{w_1, \ldots, w_n\}) \subset W$. We then know that we can write each w_i as a linear combination of in terms of the t_j's:

$$w_i = \sum_{j=1}^{m} \alpha_{j,i} t_j$$

(Let this be a definition for $\alpha_{j,i}$.) We then take $\varphi : V \to \mathrm{Span}(\{w_1, \ldots, w_n\}) \subset W$ to be the linear map with the matrix:

$$\varphi = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & & \alpha_{2,n} \\ \vdots & & \ddots & \vdots \\ \alpha_{m,1} & \alpha_{m,2} & \cdots & \alpha_{m,n} \end{bmatrix}$$

when written with respect to the bases $\{v_1, \ldots, v_n\}$ and $\{t_1, \ldots, t_m\}$. We easily verify that

$$\varphi(v_i) = \sum_{j=1}^{m} \alpha_{j,i} t_j = w_i$$

as required.

Conversely, suppose φ' satisfied $\forall i, w_i = \varphi'(v_i)$. Then the clearly $\text{Im}(\varphi') = \text{Span}(\varphi'(\{v_1, \ldots, v_n\}) = \text{Span}(\{w_1, \ldots, w_n\})$, thus we can write the matrix for $\varphi' : V \to \text{Span}(\{w_1, \ldots, w_n\})$ with respect to the bases $\{v_1, \ldots, v_n\}$ and $\{t_1, \ldots, t_m\}$. We find that if $\varphi'(v_1) = w_1$, then the ith column of this matrix must be $\begin{bmatrix} \alpha_{1,i} \\ \vdots \\ \alpha_{m,i} \end{bmatrix}$, and thus

$$\varphi' = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & & \alpha_{2,n} \\ \vdots & & \ddots & \vdots \\ \alpha_{m,1} & \alpha_{m,2} & \cdots & \alpha_{m,n} \end{bmatrix} = \varphi$$

proving φ's uniqueness.

NOTE: Why are we treating φ and φ' as maps $V \to \text{Span}(\{w_1, \ldots, w_n\})$ rather than $V \to W$? Well, we want to write matrices for φ and φ', and this requires that we have a finite basis (infinite dimensional matrices don't really make sense without a bunch of extra requirements like having only finitely many entries in a single column be nonzero, etc.), and we don't know that this is the case for W. We do something similar – this time with V – in the "spanning" part of the backwards direction, below.

$\Leftarrow$. (Assume that for any $(w_1, \ldots, w_n) \in W^n$, there exists a unique $\varphi : V \to W$ such that $\forall i, w_i = \varphi(v_i)$ in order to prove that $\{v_1, \ldots, v_n\}$ is a basis of V.) In order to prove that $\mathfrak{V} := \{v_1, \ldots, v_n\}$ is a basis, we must prove that it is linearly independent and spans.

Linear Independence. Suppose, to the contrary, that there exist α_i, not all zero, with

$$\sum_{i=1}^{n} \alpha_i v_i = 0$$

Then there must be some i_0 such that $\alpha_{i_0} \neq 0$. We then select $w_{i_0} = 1$ and $w_{j \neq i_0} = 0$ and take the corresponding φ. We then discover that

$$\begin{aligned} 0 &= \varphi(0) \\ &= \varphi\left(\sum_{i=1}^{n} \alpha_i v_i\right) \\ &= \sum_{i=1}^{n} (\alpha_i \varphi(v_i)) \\ &= \sum_{i=1}^{n} \alpha_i w_i \\ &= \alpha_{i_0} 1 \neq 0 \end{aligned}$$

which is a contradiction.

Spanning. Let x and y be distinct elements of W. Now, suppose, contrary to the claim, that there exist $v \in V$ such that $v \notin \text{Span}(\mathfrak{V})$. We know that $\mathfrak{V}$ is linearly independent (we just proved it, above), so $\{v\} \cup \mathfrak{V}$ must be linearly independent too. Consequently, $\{v\} \cup \mathfrak{V}$ is a basis of $\text{Span}(\{v\} \cup \mathfrak{V})$. Thus, by the forward direction of this problem, we can pick linear maps $\varphi_1, \varphi_2 : \text{Span}(\{v\} \cup \mathfrak{V}) \to W$ with $\forall i, \varphi_1(v_i) = \varphi_2(v_i) = w_i$, but $\varphi_1(v) = x$ and $\varphi_2(v) = y$. Because $\{v\} \cup \mathfrak{V}$ is linearly independent, we can extend it to a basis of V, and we can simultaneously extend φ_1 and φ_2 by mapping every additional basis vector to 0. We then have two distinct linear maps, the extended φ_1 and φ_2, taking v_i to w_i for all i.

□

Problem 4.

Claim 4.1. Let V and W be subspaces of $\mathbb{R}^n$. Define $V+W := \{v+w \mid v \in V, w \in w\}$. Then $\dim(V+W) = \dim(V) + \dim(W) = \dim(V \cap W)$.

Because $V \cap W$ is a subspace of $\mathbb{R}^n$, we can take a basis, say $X := \{x_1, \ldots, x_a\}$ (where $a = \dim(V \cap W)$). Because X is a linearly independent subset of V, we can extend it to a basis of V, say $\{x_1, \ldots, x_a, y_1, \ldots, y_b\}$ (where $b = \dim(V) - \dim(V \cap W)$). Similarly, we can extend $\{x_1, \ldots, x_a\}$ to a basis, $\{x_1, \ldots, x_a, z_1, \ldots, z_c\}$ of W (where $c = \dim(W) - \dim(V \cap W)$). I claim that $B := \{x_1, \ldots, x_a, y_1, \ldots, y_b, z_1 \ldots, z_c\}$ is a basis of $V+W$. If it is, then our claim is proven, for B has

$$\begin{aligned} a+b+c &= \dim(V \cap W) + (\dim(V) - \dim(V \cap W)) + (\dim(W) - \dim(V \cap W)) \\ &= \dim(V) + \dim(W) - \dim(V \cap W) \end{aligned}$$

elements.

Linear Independence. Now, assume, contrary to the claim, that there exist α_i, β_i, and γ_i, not all zero, such that

$$\left(\sum_{i=1}^{a} \alpha_i x_i\right) + \left(\sum_{i}^{b} \beta_i y_i\right) + \left(\sum_{i=1}^{c} \alpha_i z_i\right) = 0$$

Then clearly, at least one β is nonzero (for we know that $\{x_1, \ldots, x_a, z_1 \ldots, z_c\}$ is linearly independent). We can rewite the equation above as:

$$\left(\sum_{i}^{b} \beta_i y_i\right) = -\left(\sum_{i=1}^{a} \alpha_i x_i\right) - \left(\sum_{i=1}^{c} \alpha_i z_i\right)$$

Then the LHS is a nonzero element of $\text{Span}(\{y_1, \ldots, y_b\}) = V \setminus W$, and the RHS is an element of $\text{Span}(\{x_1, \ldots, x_a, z_1, \ldots, z_c\}) = W$. Thus we have a nonzero element in the intersection of $V \setminus W$ and W, which is a contradiction.

Spanning. Let $p \in V+W$. Then $p = v+w$ for some $v \in V$ and $w \in W$. v can be written as a linear combination of elements of $\{x_1, \ldots, x_a, y_1, \ldots, y_b\} \subset B$, and w can be written as a linear combination of elements of $\{x_1, \ldots, x_a, z_1, \ldots, z_c\} \subset B$, and thus their sum, p, can be written as the sum of these two linear combination and is therefore in the span of B.

□