Problem Set 7, Part B Solution Set

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1. Problem 2.5.15. Let $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ be linerar transformations.

   (a) Show there exists $S : \mathbb{R}^n \to R^n$ such that $T_1 = S \circ T_2$ if and only if $\ker T_2 \subset \ker T_1$.

   Solution. ($\Rightarrow$) Suppose there is a map S such that $T_1 = S \circ T_2$. We can quickly check that when S is restricted to values in $\operatorname{im} T_2$, it is a linear map. From this, it follows that $S(0) = 0$. We want to show $\ker T_2 \subset \ker T_1$. Let $x \in \ker T_2$. Then

   $$T_1(x) = S(T_2(x)) = S(0) = 0,$$

   and so $x \in \ker T_1$.

   ($\Leftarrow$) Now suppose $\ker T_2 \subset \ker T_1$. We define S in two steps. First, consider $y \in \operatorname{im} T_2$. This means there exists x such that $T_2(x) = y$. Then set $S(y) = T_1(x)$. We must also define S for $y \notin \operatorname{im} T_2$ (many people omitted this detail—it is very important; otherwise we have not defined S in *all* of $\mathbb{R}^n$). For $y \notin \operatorname{im} T_2$ we can define S to be whatever we please. So let's just say $S(y) = 0$ in this case.

   We need to check that S is a well-defined map. That is, we must show that if $T_2(x_1) = T_2(x_2) = y$, then $T_1(x_1) = T_1(x_2)$. Indeed, if $T_2(x_1) = T_2(x_2)$, then $T_2(x_1 - x_2) = 0$, so $x_1 - x_2 \in \ker T_2$. But we know $\ker T_2 \subset \ker T_1$, so $x_1 - x_2 \in \ker T_1$. Hence $T_1(x_1) = T_1(x_2)$. □

   (b) Show that there exists $S : \mathbb{R}^n \to R^n$ such that $T_1 = T_2 \circ S$ if and only if $\operatorname{im} T_1 \subset \operatorname{im} T_2$.

   Solution. ($\Rightarrow$) Suppose there is a map S such that $T_1 = T_2 \circ S$. We want to show $\operatorname{im} T_1 \subset \operatorname{im} T_2$. Let $x \in \operatorname{im} T_1$. This means there exists y such that $T_1(y) = x$. But $T_1(x) = T_2 \circ S(x) = T_2(S(x))$, and since $S(x) \in \mathbb{R}^n$, we have $y \in \operatorname{im} T_2$.

   ($\Leftarrow$) Now suppose $\operatorname{im} T_1 \subset \operatorname{im} T_2$. We'll construct the desired map S. For any $x \in \mathbb{R}^n$ we have $T_1(x) \in \operatorname{im} T_1$, so $T_1(x) \in \operatorname{im} T_2$, *i.e.*, there is at least one $y_k \in R^n$ such that $T_1(x) = T_2(y_k)$. Among the y_k choose y to be the one with smallest norm. If there is more than one such y_k, then choose the one whose first component is smallest, and so on. Set $S(x) = y$. This way, S is well-defined, and $T_1(x) = T_2 \circ S(x)$ for all $x \in \mathbb{R}^n$. □

2. Problem 2.5.16

   (a) Find a polynomial of degree 2 $p(x) = a + bx + cx^2$ such that $p(0) = 1$, $p(1) = 4$ and $p(3) = -2$.

Solution. Almost no-one had trouble with this. We get $p(x) = 1 + 5x - 2x^2$. $\square$

(b) Show that if $x_0, \dots, x_n$ are $n+1$ distinct points in $\mathbb{R}$, and $a_0, \dots, a_n$ are any numbers, there exists a unique polynomial of degree n such that $p(x_i) = a_i$ for each $i = 0, \dots, n$.

Solution I. Consider the map $M : P_n \to \mathbb{R}^{n+1}$, where

$$p \mapsto \begin{bmatrix} p(x_0) \\ \vdots \\ p(x_n) \end{bmatrix}.$$

We can easily check that M is a linear map. We want to show that the map M is surjective; this will prove that the desired polynomial of degree n exists. By the rank-nullity theorem,

$$\dim(\operatorname{im} M) + \dim(\ker M) = n + 1,$$

So to prove surjectivity, it is enough to show $\dim(\ker M) = 0$, or $\ker M = \{0\}$ (here 0 denotes the zero polynomial). Let p be a polynomial in $\ker M$. Then we must have $p(x_i) = 0$ for $i = 0, \dots, n$. But then p must be a degree n poynomial with $n+1$ distinct roots. This can only happen if p is the zero polynomial. Hence $\ker M = \{0\}$. To show p is unique, we need the map M to be injective. But we've already seen that $\ker M = \{0\}$, so we're done. We killed two birds with one stone. $\square$

Solution II (Lagrange interpolation). This problem can also be done using Lagrange interpolation. This approach is nicer in some ways if you are a constructivist. This solution actually tells you what p looks like. Consider the monomial

$$m_i(x) = \frac{(x - x_1) \cdots (x - x_n)}{(x_i - x_1) \cdots (x_i - x_n)},$$

where the term $(x_i - x_i)$ is omitted in the denominator of $m_i(x)$ for obvious reasons. We have

$$m_i(x_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $p(x) = \displaystyle\sum_{i=0}^{n} a_i m_i(x)$. We have $p(x_i) = a_i$ for $i = 0, \dots, n$, as required. So p exists. Why is p unique? Suppose there are two polynomials p and q that satisfy the given conditions. Then $r = p - q$ would be a polynomial of degree at most n with at least $n+1$ distinct roots, unless r is the zero polynomial. $\square$

(c) Let x_i and a_i be as above, and let $b_0, \dots, b_n$ be some further set of numbers. Find a number k such that there exists a unique polynomial of degree k with $p(x_i) = a_i$ and $p'(x_i) = b_i$, for $i = 0, \dots, n$.

Solution (due to Kevin Weil and Minhua Zhang). We will show that $k = 2n + 1$. This time, construct a map $M : P_k \to \mathbb{R}^{2n+2}$ given by

$$p \mapsto \begin{bmatrix} p(x_0) \\ \vdots \\ p(x_n) \\ p'(x_0) \\ \vdots \\ p'(x_n) \end{bmatrix}.$$

We'll show that for $k = 2n + 1$ the map M is bijective. Since the dimensions of P_{2n+1} and $\mathbb{R}^{2n+2}$ are equal, it will suffice to prove the map is injective, *i.e.*, $\ker M = \{0\}$. Let $p \in \ker M$. Then $p'(x_i) = 0$ for $i = 0, \ldots, n$. This means that the x_i are at least double roots of p (this is an easy consequence of the fundamental theorem of algebra and the fact that the real numbers have characteristic zero). Hence, p has at least $2n+2$ roots, but is a polynomial of degree $2n+1$. Again, p must be the zero polynomial. Thus $\ker M = \{0\}$. We are done! ☐