Math 25a Solution Set #7 (Part C)

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Problem 1

Let V be a vector space with basis $S = \{v_1, \ldots, v_n\}$ and W a vector space with basis $B = \{w_1, \ldots, w_m\}$. Also let $L : V \to W$ be a linear map with matrix A with respect to the two bases.

Part (a). We need to show that $B_1 = \{w_1, \ldots, w_{i-1}, \frac{1}{\alpha}w_i, w_{i+1}, \ldots, w_m\}$, for any$1 \leq i \leq m$ and $\alpha \neq 0$ is again a basis of W. Since B_1 has m elements, to show it is a basis for an m-dimensional vector space W, it is enough to check that it spans it, or that it is LI. Any one of the two suffices.

So let's show B_1 spans W. Pick any $w \in W$. Since $B = w_1, \ldots, w_m$ is a basis for W, w is a linear combination of elements of B, so that $\exists a_1, \ldots, a_m \in \mathbb{R}$ such that $w = a_1w_1 + \ldots + a_mw_m$. Therefore we have:

$$w = a_1w_1 + \ldots + a_{i-1}w_{i-1} + \alpha a_i \left(\frac{1}{\alpha}w_i\right) + a_{i+1}w_{i+1} \ldots + a_mw_m$$

We conclude $\forall w \in W$, w is in the span of B_1 and hence B_1 spans W.

As for determining the new matrix A_1 of L with respect to bases S of V and B_1 of W, the only thing to keep in mind is that columns of A_1 are images of the basis vectors of V under L, represented as linear combinations of elements of B_1.

The above proof that B_1 is a basis gives us the recipe of turning a linear combination of elements of B into a linear combination of elemeents of B_1 - just multiply the i-th coordinate by α. So A_1 will be a matrix obtained by multiplying the i-th row of A by α and leaving all other entries the same. This can be fancily written using elementary matrices as $A_1 = E_1(i, \alpha)A$, since A_1 is obtained from A by an eleementary row operation. ∎

Part (b). We need to show that $B_2 = \{w_1, \ldots, w_{i-1}, w_i - \beta w_j, w_{i+1}, \ldots, w_m\}$, for any$1 \leq i \neq j \leq m$ and $\beta \in \mathbb{R}$ is again a basis of W. As in the previous problem, we notice that the fact that B_2 has m elements makes our job easier, since proving that it spans W is again sufficient. For any $w \in W$, we have

$\exists a_1, \ldots, a_m \in \mathbb{R}$ such that $w = a_1 w_1 + \ldots + a_m w_m$. Then, we have:

$$\begin{aligned} w &= a_1 w_1 + \ldots + a_{j-1} w_{j-1} + (a_j + \beta a_i) w_j + \\ &+ a_{j+1} w_{j+1} + \ldots + a_{i-1} w_{i-1} + a_i (w_i - \beta w_j) + a_{i+1} w_{i+1} + \ldots + a_m w_m \end{aligned}$$

with the order in which indices i and j appear in the formula above being inessential. We have written w as a linear combination of elements of B_2, so we conclude B_2 spans W, and hence is a basis.

The new matrix A_2 of L is again determined by observing that if some vector in W had coordinates $(a_1, \ldots, a_m)$ with respect to B, its coordinates with respect to B_2, as we have seen above, will be $(a_1, \ldots, a_{j-1}, a_j + \beta a_i, a_{j+1}, \ldots, a_m)$, so that A_2 is obtained by multiplying the i-th row of A by β and adding it to the j-th row, with all entries of A remaining the same. Equivalently, A_2 is obtained from A by an elementary row operation: $A_2 = E_2(i, j, \beta) A$. ∎

Part (c). If A' is any matrix obtained from A by row reduction, we need to show there exists a basis $B' = \{w'_1, \ldots, w'_m\}$ of W such that A' is the matrix of L with respect to $S = \{v_1, \ldots, v_n\}$ and $B' = \{w'_1, \ldots, w'_m\}$.

Having solved the previous two problems, most people had no trouble noticing that (more than) half the job has been done already. The main problem here was to recall the previously proven statement that row switches can be written as a composition of the other two row operations, so that any row reduction can be performed using just the following two row operations: (1) - multiplying a row by a scalar, and (2) - adding a scalar multiple of one row to another. As we have seen above, row operation (1) corresponds to the change of basis described in part (a), whereas (2) is taken care of in (b).

At each step of the row reduction we perform the corresponding operation on the basis of W, and as parts (a), (b) show we still have a matrix of L with respect to the new basis, after each step. After all row operations and corresponding basis changes are performed, we end up with matrix A' and some new basis B' of W such that A' is a matrix of L with respect to S and B'. ∎

Part (d). In this part of the problem, an inductive construction of B' was described, as follows. Let $i_1 > 0$ be the smallest index such that $L(v_{i_1}) \neq 0$ and define $w'_1 = L(v_{i_1})$. Let $i_2 > i_1$ be the smallest index such that $L(v_{i_2}) \notin Span(w'_1)$ and define $w'_2 = L(v_{i_2})$. Inductively, let $i_k > i_{k-1}$ be the smallest index such that $L(v_{i_k}) \notin Span(w'_1, \ldots, w'_{k-1})$ and define $w'_k = L(v_{i_k})$. This way we get linearly independent $\{w'_1, \ldots, w'_p\}$, which can be extended to a basis $B' = \{w'_1, \ldots, w'_p, \ldots, w'_m\}$ of W. We need to show the matrix A' of L with respect to S and B' is in echelon form.

If we let $[A']_{ij} = a'_{ij}$, using the fact that A' is a matrix of L with respect to S, B', and keeping in mind that the i-th column of A' is $L(v_i)$ written in basis B', we get that $\forall i$ such that $1 \leq i \leq n$,

$$L(v_i) = a_{1,i} w'_1 + \ldots + a_{m,i} w'_m$$

If our i in the above formula is i_k for some $k \in \{1, \ldots, p\}$, we know that $w'_k = L(v_{i_k})$, so that the i_k-th column of A' has $m-1$ entries equal 0, and only one, $a_{k,i_k} = 1$. Since i_k was chosen as the smallest index such that $L(v_{i_k}) \notin Span(w'_1, \ldots, w'_{k-1})$, all columns before the i_k-th are in this span, hence have zeros starting from the k-th entry, so that a_{k,i_k} is the first nonzero entry in its row, and the i_k-th column is pivotal. This holds $\forall k \in 1, \ldots, p$.

If our i in the formula above is not i_k for any k, then it has to be in the span of those columns preceding it which are i_k for some k (for if not, it would be chosen as the "next" i_k), so that we get a non-pivotal column. Specifically, this implies all columns after the i_p-th (if any such exist) have zero entries below the p-th row, so that the rows below p-th are zero. It is easily verified that this suffices to conclude A' is in echelon form, since all four conditions hold:

(1) In every row the first nonzero entry is 1. For the first p rows this was shown explicitly. Rows below them are zero so the statement holds vacuously.

(2) The pivotal 1 of a lower row is always to the right of the pivotal 1 of a higher row, since our construction of B' was such that $k_1 > k_2 \Rightarrow \mathrm{i}_{k_1} > i_{k_2}$.

(3) In every column that contains a pivotal 1, all other entries are 0, a property we have shown holds for all columns of index i_k for some k.

(4) Any rows consisting entirely of zeros are at the bottom, exactly! ∎

Part (e). This problem asked you to prove a "converse" of the construction above, i.e. if a matrix A' of L with respect to bases $S = \{v_1, \ldots, v_n\}$ of V and $B' = \{w'_1, \ldots, w'_m\}$ of W is in echelon form, then B' was constructed as in (d). Most people did this correctly, but a large majority misinterpreted the last sentence of this question. Even if you are in the lucky minority, please reread this sentence, and look again at the construction of B'. This construction gives you (via an iterative algorithm) $\{w'_1, \ldots, w'_p\}$ as a function of L and $\{v_1, \ldots, v_n\}$, but then it says "Now extend this to a basis of W...!" In general, if $p < m$, there are many ways to extend a given LI set to a basis, and hence our B' is NOT unique, only the first p basis vectors are!

After due tirade, for the proof. We denote the indices of pivotal columns of A' with $i_1, \ldots, \imath_p$, in the order they appear in A' from left to right. The third property of echelon form gives us $L(v_{i_k}) = w'_k$. The first property tells us that $L(v_{i_k})$ is the leftmost column of A' such that it is not in the span of the previous ones, which shows that the choice of i_k's is precisely the one described by the construction in (d). ∎

Part (f). The misunderstanding the last sentence of part (d) caused induced some people to (falsely) conclude that echelon form is unique because B', as constructed in part (d), is. There is a bit more work here, after all (but not too much). So let L be the lin. transf. with matrix A with respect to standard bases on V, W. Given L and this basis on V, let $\{w'_1, \ldots, w'_p\}$ be the uniquely determined vectors obtained by the procedure in (d).

Since A' was obtained from A by row reduction, part (c) gives us that A' is a matrix of L with respect to some basis S, B_0 of V and W resp. But now we can apply part (e) to this matrix of L to conclude that, since it is in echelon form, B_0 must have been obtained by extending $\{w'_1, \ldots, w'_p\}$ (defined in the prev. paragraph) to a basis of W. Since the i-th column of A' gives $L(V_i)$ in terms of elements of B_0, the first p top entries in every column are uniquely determined, which shows that the top p rows of A' are uniquely determined by A. But the construction in part (d) shows that all rows below the p-th are zero, so that in particular, they do not depend on $\{w'_{p+1}, \ldots, w'_m\}$. ∎

Problem 2

Part (a). Please see the problem set for the defintion and properties of exact sequences. The hint that urged you to rely on the rank-nullity theorem seemed to be sufficient for most people, but here is the whole proof (courtesy of Arthur Baum!):

In a short exact sequence, we know that $Img(d_2) = Ker(d_3) = V_3$, that $Img(d_1) = Ker(d_2)$, and that $Ker(d_1) = Img(d_0) = 0$. Furthermore, we know that $\dim[V_1] = \dim[Img(d_1)] + \dim[Ker(d_1)]$ and that $\dim[V_2] = \dim[Img(d_2)] + \dim[Ker(d_2)]$. From these facts, it follows that

$$\begin{aligned}
\dim V_2 &= \dim Img(d_2) + \dim Ker(d_2) \\
\dim V_2 &= \dim V_3 + \dim Img(d_1) \\
\dim V_2 &= \dim V_3 + \dim V_1 - \dim Ker(d_1) \\
\dim V_2 &= \dim V_3 + \dim V_1 - 0 \\
\dim V_2 &= \dim V_1 + \dim V_3
\end{aligned}$$

∎

Part (b). We will now show for an arbitrary exact sequence that

$$\sum_i (-1)^i \dim V_i = 0.$$

If we write the sum out term by term, it is equal to

$$-(\dim[Ker(d_1)] + \dim[Img(d_1)]) + (\dim[Ker(d_2)] + \dim[Img(d_2)]) -$$

$$(\dim[Ker(d_3)] + \dim[Img(d_3)]) + \ldots + (-1)^k (\dim[Ker(d_k)] + \dim[Img(d_k)]).$$

Note that $Img(d_1) = Ker(d_2)$, $Img(d_2) = Ker(d_3)$, and so on. If we rearrange the terms above, we have

$$-\dim[Ker(d_1)] + (-\dim[Img(d_1)] + \dim[Ker(d_2)]) +$$

$$(\dim[Img(d_2)] - \dim[Ker(d_3)]) + \ldots + (-1)^k (-\dim[Img(d_{k-1})] +$$
$$\dim[Ker(d_k)]) + (-1)^k \dim[Img(d_k)].$$

Since all but the first and last terms equal zero, the Euler characteristic of the sequence is

$$-\dim[Ker(d_1)] + (-1)^k \dim[Img(d_k)] = -0 \pm 0 = 0.$$

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