MATH 25A – PROBLEM SET #8
DUE WEDNESDAY NOVEMBER 21

1. Part A

(1) Prove that the following are equivalent for an $n \times n$ matrix A:
- A has an inverse.
- The columns of A are linearly independent.
- The rows of A are linearly independent.

(2) Problems 2.7.3, 2.7.4(a), 2.7.14(a),(b).

2. Part B

(1) Problem 2.9.1.

(2) Problem 2.7.13. Replace the matrix given there by the identity matrix I_2. In other words: does every matrix close to the identity I_2 have a square root?

3. Part C

(1) Recall that an inner product $< \cdot,\cdot >$ on a vector space is a bilinear, symmetric function $V \times V \to \mathbb{R}$ satisfying the positivity condition: $< v,v > \geq 0$ for all $v \in V$ and $< v,v >= 0$ if and only if $v = 0$.

(a) Prove that
$$< f(x), g(x) >= \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$$
defines an inner product on $\mathbb{R}[x]$, the space of polynomials in x.

(b) Let $T_n(x) = \cos(n \arccos(x))$ for $n = 0, 1, 2, \ldots$. Prove that
$$T_{n+1} = 2xT_n(x) - T_{n-1}(x).$$
Deduce that T_n is a polynomial for all n. The polynomials T_n are called Chebyshev polynomials. (Hint: to prove the formula, find a relation between $\cos(a+b), \cos(a-b), \cos(a)$ and $\cos(b)$.)

(c) Prove that T_n for $n = 0, 1, 2, \ldots$ form an orthogonal basis of $\mathbb{R}[x]$ with respect to the inner product given above.

(2) Recall that the dual V^* of a vector space V is
$$V^* = Lin(V, \mathbb{R}).$$
Let $< \cdot,\cdot >$ be an inner product on a vector space V.

(a) Prove that we have a linear map $L : V \to V^*$, defined by
$$L(v)(w) =< v, w > .$$
(Note: here $L(v)$ is an element of V^*, hence a function on V, and it can be applied to an element $w \in V$. You have to prove first that the map L is well-defined, that means $L(v)$ is a linear function $V \to \mathbb{R}$, and second that the map L is linear.)

(b) Prove that if V is finite dimensional, then L is an isomorphism. (Hint: what is the dimension of V^* ?)

(c) Let P_d be the set of polynomials in x of degree d or less. Prove that there exists a unique polynomial $p(x) \in P_d$ such that for any $q(x) \in P_d$ we have

$$\int_0^1 p(x)q(x)dx = q(3)$$