Problem Set 8, Part C Solution Set

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1. Recall that an inner product $\langle \cdot, \cdot \rangle$ on a vector space V is a bilinear, symmetric function $V \times V \to \mathbb{R}$ satisfying the positivity condition: $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.

   (a) Prove that

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx$$

   defines an inner product on $\mathbb{R}[x]$, the space of polynomials in x.

   Solution (Adapted from John Provine's Solution Set). We must show that $\langle f(x), g(x) \rangle$ satisfies positivity, nondegeneracy, symmetry, and bilinearity.

   - *Positivity.* We want $\langle f(x), f(x) \rangle \geq 0$ for all $f(x) \in \mathbb{R}[x]$. Since $[f(x)]^2 / \sqrt{1-x^2} \geq 0$ for all $f(x) \in \mathbb{R}[x]$ and $-1 < x < 1$, it follows that

$$\langle f(x), f(x) \rangle = \int_{-1}^{1} \frac{[f(x)]^2}{\sqrt{1-x^2}}\, dx \geq 0.$$

   - *Nondegeneracy.* $\langle f(x), f(x) \rangle = 0$ if and only if $f(x) = 0$. Suppose that $f(x) = 0$, then clearly

$$\langle f(x), f(x) \rangle = \int_{-1}^{1} 0\, dx = 0.$$

     Conversely, suppose that $\langle f(x), f(x) \rangle = \int_{-1}^{1} \left([f(x)]^2 / \sqrt{1-x^2} \right) dx = 0$. Recall the integrand is non-negative in the interval $[-1, 1]$. Moreover, if there is a point x in the interval for which $|f(x)| > 0$, then there is a small interval around x for which $|f| > 0$ since f is a polynomial and polynomials are continuous. But then there would be a non-zero area under the graph of $[f(x)]^2 / \sqrt{1-x^2}$, so that $\int_{-1}^{1} \left([f(x)]^2 / \sqrt{1-x^2} \right) dx > 0$. Hence $f(x) \equiv 0$.

   - *Symmetry.* $\langle f(x), g(x) \rangle = \langle g(x), f(x) \rangle$ for all $f(x), g(x) \in \mathbb{R}[x]$. From the commutativity of multiplication in $\mathbb{R}[x]$, we have

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx = \int_{-1}^{1} \frac{g(x)f(x)}{\sqrt{1-x^2}}\, dx = \langle g(x), f(x) \rangle .$$

- *Bilinearity.* For all $f(x), g(x), h(x) \in \mathbb{R}[x]$ and $\alpha, \beta \in \mathbb{R}$,

  $$\langle \alpha f(x) + \beta g(x), h(x) \rangle = \alpha \langle f(x), h(x) \rangle + \beta \langle g(x), h(x) \rangle ,$$

  and

  $$\langle f(x), \alpha g(x) + \beta h(x) \rangle = \alpha \langle f(x), g(x) \rangle + \beta \langle f(x), h(x) \rangle .$$

  Indeed,

  $$\begin{aligned} \langle \alpha f(x) + \beta g(x), h(x) \rangle &= \int_{-1}^{1} \frac{(\alpha f(x) + \beta g(x))\, h(x)}{\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1} \frac{\alpha f(x)h(x) + \beta g(x)h(x)}{\sqrt{1-x^2}} dx \\ &= \int_{-1}^{1} \frac{\alpha f(x)h(x)}{\sqrt{1-x^2}} dx + \int_{-1}^{1} \frac{\beta g(x)h(x)}{\sqrt{1-x^2}} dx \\ &= \alpha \int_{-1}^{1} \frac{f(x)h(x)}{\sqrt{1-x^2}} dx + \beta \int_{-1}^{1} \frac{g(x)h(x)}{\sqrt{1-x^2}} dx \\ &= \alpha \langle f(x), h(x) \rangle + \beta \langle g(x), h(x) \rangle . \end{aligned}$$

  Since $\langle \cdot, \cdot \rangle$ is symmetric, we can write

  $$\begin{aligned} \langle f(x), \alpha g(x) + \beta h(x) \rangle &= \langle \alpha g(x) + \beta h(x), f(x) \rangle \\ &= \alpha \langle g(x), f(x) \rangle + \beta \langle h(x), f(x) \rangle \\ &= \alpha \langle f(x), g(x) \rangle + \beta \langle f(x), h(x) \rangle . \end{aligned}$$

Hence $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{R}[x]$. □

(b) Let $T_n(x) = \cos(n \arccos(x))$ for $n = 0, 1, 2, \ldots$. Prove that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Deduce that T_n is a polynomial for all n. The polynomials T_n are called *Chebyshev polynomials.*

Solution (Adapted from John Provine's Solution Set). We have

$$\begin{aligned} T_{n+1}(x) + T_{n-1}(x) &= \cos((n+1)\arccos(x)) + \cos((n-1)\arccos(x)) \\ &= \cos(n \arccos(x)) \cos(\arccos(x)) \\ &\quad - \sin(n \arccos(x)) \sin(\arccos(x)) \\ &\quad + \cos(n \arccos(x)) \cos(\arccos(x)) \\ &\quad + \sin(n \arccos(x)) \sin(\arccos(x)) \\ &= 2\cos(n \arccos(x)) \cos(\arccos(x)) \\ &= 2xT_n(x). \end{aligned}$$

Hence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for all $n > 1$, as desired. It follows that $T_n(x)$ is a polynomial for all n. We'll use 'strong' induction. First, note that $T_0(x) = 1$ and

$T_1(x) = x$ are polynomials. Now assume that $T_n(x)$ is a polynomial for all positive integers up to and including some n, then we can write

$$T_{n-1}(x) = \sum_i a_i x^i \quad \text{and} \quad T_n(x) = \sum_i b_i x^i$$

for some $a_i, b_i \in \mathbb{R}$. Thus

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) = 2x \sum_i b_i x^i - \sum_i a_i x^i = \sum_i c_i x^i$$

for some $c_j \in \mathbb{R}$, which means that $T_{n+1}(x)$ is also a polynomial. Hence $T_n(x)$ is a polynomial for all non-negative integers n. □

(c) Prove that T_n for $n = 0, 1, 2, \ldots$ form an orthogonal basis of $\mathbb{R}[x]$ with respect to the inner product given above.

Solution. We'll start off by showing the T_n are orthogonal. Indeed,

$$\langle T_m(x), T_n(x) \rangle = \int_{-1}^{1} \frac{\cos(m \arccos(x)) \cos(n \arccos(x))}{\sqrt{1-x^2}}\, dx.$$

Now make the subtitution $u = \arccos(x)$ to get

$$\begin{aligned}
\int_{-1}^{1} \frac{\cos(m \arccos(x)) \cos(n \arccos(x))}{\sqrt{1-x^2}}\, dx &= -\int_0^{\pi} \cos(mu)\cos(nu)\, du \\
&= -\frac{1}{2}\int_0^{\pi} \cos((m+n)u) + \cos((m-n)u)\, du \\
&= -\frac{1}{2}\left(\frac{\sin((m+n)u)}{m+n} + \frac{\sin((m-n)u)}{m-n} \right)\Bigg|_0^{\pi} \\
&= 0,
\end{aligned}$$

for integers m, n such that $m \neq n$. If $m = n$ then

$$-\int_0^{\pi} \cos(mu)\cos(nu)\, du = -\int_0^{\pi} \cos^2(nu)\, du = -\frac{1}{2}\int_0^{\pi} 1 + \cos(2nu)\, du = \frac{\pi}{2}$$

Now we'll show the Chebyshev polynomials are a basis for $\mathbb{R}[x]$. First, they are linearly independent because they are orthogonal (see the remark below). To show they span $\mathbb{R}[x]$ we verify that $\{1, x, x^2, \ldots\} \subset \text{Span}\{T_0, T_1, \ldots\}$. The recursion formula from part (b) tells us that T_n is a polynomial of degree n. Say $T_n(x) = a_n x^n + \cdots + a_0$ and $T_{n-1} = b_{n-1}x^{n-1} + \cdots + b_0$. Then

$$\frac{1}{a_n} T_n - \frac{a_{n-1}}{b_{n-1}} T_{n-1} = x^n + c_{n-2}x^{n-2} + \cdots + c_0.$$

Notice that a_n and b_{n-1} are non-zero. Now repeat this procedure. Use T_{n-i} to get rid of x^{n-i} like we have shown above for $i = 2, \ldots, n$. We see that $x^n \in \text{Span}\{T_0, T_1, \ldots\}$ for any non-negative integer n. Thus, $\{T_n\}$ is a basis for $\mathbb{R}[x]$. □

Remark. I noticed there was some confusion as to what it means for an infinite set of vectors to be linearly independent. Such a set is linearly independent if any finite subset of it is linearly independent in the good-old-fashioned way.

Also, many of you did not notice/remember the fact that orthogonality gives you linear independence. So let's prove that here quickly. Suppose we have an inner-product space V and a collection of vectors $\{v_\alpha\}_{\alpha \in I}$, (which might be infinite depending on the indexing set I) such that $\langle v_\alpha, v_\beta \rangle = 0$ whenever $\alpha \neq \beta$ and $\langle v_\alpha, v_\alpha \rangle \neq 0$. Then if we had a relation of the form

$$a_1 v_{\alpha_1} + \cdots + a_n v_{\alpha_n} = 0,$$

we could take the inner product of both sides of the equation to get

$$\begin{aligned} a_1 \langle v_{\alpha_1}, v_{\alpha_k} \rangle + \cdots + a_n \langle v_{\alpha_n}, v_{\alpha_k} \rangle &= \langle 0, v_{\alpha_k} \rangle \\ \Rightarrow a_k &= 0 \quad \forall\, k. \end{aligned}$$

2. Recall that the dual space V^* of a vector space V is

$$V^* = Lin(V, \mathbb{R})$$

Let $\langle \cdot, \cdot \rangle$ be an inner product of the vector space V.

(a) Prove that we have a linear map $L : V \to V^*$, defined by

$$L(v)(w) = \langle v, w \rangle.$$

Solution. First, we need to show that L is well defined, i.e., that the image of $L(v)$ is an element of V^* (or better said, that $L(v)$ is a linear map $V \to \mathbb{R}$). This is easy; we just have to use the linearity of the inner product,

$$\begin{aligned} L(v)(w_1 + w_2) &= \langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle = L(v)(w_1) + L(v)(w_2) \\ L(v)(\alpha w) &= \langle v, \alpha w \rangle = \alpha \langle v, w \rangle = \alpha L(v)(w) \quad \forall\, \alpha \in \mathbb{R}, v \in V. \end{aligned}$$

Now we just need to show that L is a linear map $V \to V^*$. Again, we just need to use the linearity of the inner product,

$$\begin{aligned} L(v_1 + v_2)(\cdot) &= \langle v_1 + v_2, \cdot \rangle = \langle v_1, \cdot \rangle + \langle v_2, \cdot \rangle = L(v_1)(\cdot) + L(v_2)(\cdot) \\ L(\alpha v)(\cdot) &= \langle \alpha v, \cdot \rangle = \alpha \langle v, \cdot \rangle = \alpha L(v)(\cdot) \quad \forall\, \alpha \in \mathbb{R}. \end{aligned}$$

□

(b) Prove that if V is finite dimensional, then L is an isomorphism.

Solution. Suppose V is finite dimensional. First, we'll show L is injective by looking at its kernel. Let v be a vector in $\ker L$. Then $\langle v, w \rangle = 0$ for all $w \in V$. In particular, $\langle v, v \rangle = 0$. Since $\langle \cdot, \cdot \rangle$ is an inner product, we must have $v = 0$. Hence $\ker L = \{0\}$ and L is injective.

Instead of directly showing L is surjective, we'll prove $\dim V = \dim V^*$. Since V is finite dimensional, applying the rank-nullity theorem, we'll get

$$\begin{aligned} \dim \operatorname{im} L + \dim \ker L &= \dim V, \\ \dim \operatorname{im} L + 0 &= \dim V \ (= \dim V^*), \\ \dim \operatorname{im} L &= \dim V^*. \end{aligned}$$

From this it is clear L will be surjective.

So all we have left to do is show $\dim V = \dim V^*$. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Define the linear maps $v_i^* \in V^*$ $(i = 1, \ldots, n)$ by

$$v_i^*(v_j) = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j. \end{cases}$$

(You should check these maps are well-defined and linear.) We'll show $\{v_1^*, \ldots, v_n^*\}$ is a basis for V^*.

First, $\{v_1^*, \ldots, v_n^*\}$ spans V^*. Indeed, take any $T \in V^*$. T is determined by its action on a set of basis vectors. Say $T(v_i) = x_i$. Define the linear map $U \in \operatorname{Span}\{v_1^*, \ldots, v_n^*\} \subset V^*$ by setting,

$$\begin{aligned} U(a_1 v_1 + \cdots + a_n v_n) &= x_1 v_1^*(a_1 v_1 + \cdots + a_n v_n) + \cdots + x_n v_n^*(a_1 v_1 + \cdots + a_n v_n), \\ &= x_1 v_1^*(a_1 v_1) + \cdots + x_n v_n^*(a_n v_n), \\ &= a_1 x_1 + \cdots a_n x_n. \end{aligned}$$

(We used the definition of the v_i^* several times in the last two equalites.) We can easily check that T and U coincide on basis vectors of V. Hence they must be the same map. So $T \in \operatorname{Span}\{v_1^*, \ldots, v_n^*\}$.

Now we'll show $\{v_1^*, \ldots, v_n^*\}$ is a linearly independent set of vectors in V^*. Suppose we have a relation of the form

$$a_1 v_1^* + \cdots + a_n v_n^* = 0.$$

Notice the left hand side of this equation is an element of V^*. The zero on the right hand side is the zero-map. So let's apply both sides of this equation to a basis vector v_j. Using the definition of the v_i^*'s we obtain

$$\begin{aligned} a_1 v_1^*(v_j) + \cdots + a_n v_n^*(v_j) &= 0(v_j), \\ a_1 \cdot 0 + \cdots + a_j \cdot 1 + \cdots a_n \cdot 0 &= 0, \\ a_j &= 0 \text{ for any } j \in \{1, \ldots, n\}. \end{aligned}$$

Hence the v_i^*'s are linearly independent. □

Remark. We have in fact shown that V and V^* are isomorphic as vector spaces. However, our isomorphism depended on an initial choice of basis. So it is not *canonical.*

(c) Let P_d be the set of polynomials in x of degree d or less. Prove that there exists a unique polynomial $p(x) \in P_d$ such that for any $q(x) \in P_d$ we have

$$\int_0^1 p(x)q(x)\,dx = q(3).$$

Solution. Recall that

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)\,dx$$

defines an inner product on P_d (we showed this is class). By part (b) the map $L : P_d \to P_d^*$ defined by

$$L(p(x))(q(x)) = \int_0^1 p(x)q(x)\,dx$$

is an isomorphism of vector spaces. Define the map $T : P_d \to \mathbb{R}$ by $T(q(x)) = q(3)$. One can easily check this is a linear map. So $T \in P_d^*$. Let $p(x)$ be the pre-image of T under L. Since L is an isomorphism, $p(x)$ is the unique polynomial such that

$$\int_0^1 p(x)q(x)\,dx = q(3).$$

□