Problem Set 9, Part A Solution Set

Tony Várilly

Math 25a, Fall 2001

1. Problem 3.1.8.

(a) For what values of a and b are the sets X_a and X_b defined by the equations

$$x - y^2 = a \quad \text{and} \quad x^2 + y^2 + z^2 = b,$$

respectively smooth surfaces in $\mathbb{R}^3$?

Solution. Define the functions F, $G : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x, y, z) = x - y^2 - a,$$
$$G(x, y, z) = x^2 + y^2 + z^2 - b,$$

so that X_a is just the level set $F = 0$ and X_b is the level set $G = 0$. For X_a to be smooth, we want $[DF(x, y, z)] \neq 0$. But

$$[DF(x, y, z)] = \begin{bmatrix} 1 & -2y & 0 \end{bmatrix},$$

Clearly, there is no triple (x, y, z) for which this matrix has all its entries vanishing simultaneously. So X_a is a smooth surface for all values of a.
Similarly, for X_b to be smooth, we want $[DG(x, y, z)] \neq 0$. This time we compute

$$[DG(x, y, z)] = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix},$$

which vanishes at the origin. At this point $b = 0$, as can be seen from the defining equation for X_b. If $b = 0$, then X_b is just a point, so it is not smooth. Note that for $b < 0$, X_b is empty, which is vacously smooth. □

(b) For what values of a and b is the intersection $X_a \cap X_b$ a smooth curve? What is the geometric relation between X_a and Y_b for other values of a and b?

Solution. Define $H : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$H(x, y, z) = \begin{bmatrix} x - y^2 - a \\ x^2 + y^2 + z^2 - b \end{bmatrix},$$

so that $X_a \cap X_b$ is the level set $H = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. In order that $X_a \cap X_b$ be a smooth curve, it is necessary that $[DH(x,y,z)]$ have maximal rank, i.e., rank 2 in this case. We compute

$$[DH(x,y,z)] = \begin{bmatrix} 1 & -2y & 0 \\ 2x & 2y & 2z \end{bmatrix}.$$

This matrix has rank at least 1 because the upper left entry is a non-zero constant. Now we try to distinguish the cases when the matrix has rank 1 and when it has rank 2, i.e, we study the possible linear dependence of its rows. Suppose α and β are real numbers such that

$$\alpha \begin{pmatrix} 1 \\ -2y \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 0.$$

We get the system of equations

$$\alpha + 2\beta x = 0, \tag{1}$$
$$2y(-\alpha + \beta) = 0, \tag{2}$$
$$2\beta z = 0. \tag{3}$$

From (3) we either have $\beta = 0$ or $z = 0$. If $\beta = 0$ then (1) tells us $\alpha = 0$, in which case, the rows of $[DH(x,y,z]$ are linearly independent and the matrix has rank 2.

If $z = 0$ then from (2) we either have $\alpha = \beta$ or $y = 0$. If $\alpha = \beta \neq 0$, then (1) tells us $x = -1/2$. In this case,

$$\left.\begin{aligned} -1/2 - y^2 = a, \\ 1/4 + y^2 + 0 = b. \end{aligned}\right\} \implies a + b + 1/4 = 0. \tag{4}$$

If $y = 0$ then we get

$$\left.\begin{aligned} x - 0 = a, \\ x^2 + 0 + 0 = b. \end{aligned}\right\} \implies a^2 = b. \tag{5}$$

In conclusion, if both (4) and (5) *do not happen*, then $[DH(x,y,z)]$ has rank 2, and $X_a \cap X_b$ is a smooth curve.

Now let us examine the geometric meaning of conditions (4) and (5) to check that indeed $X_a \cap X_b$ s not smooth for these values of a and b.

X_a is a paraboloid (it looks like a parabolic sheet) and X_b is a sphere of radius $\sqrt{b}$ centered at the origin. If $a + b + 1/4 = 0$ the paraboloid is tangent at two points to the sphere, i.e., $X_a \cap X_b$ consists of two points and is therefore not smooth.

If $a^2 = b$ and $a > 0$, the paraboloid is tangent to the sphere at the point $(a, 0, 0)$, and $X_a \cap X_b$ consists of a single point, so it is not smooth. If $a^2 = b$ and $a < 0$ then the paraboloid cuts through the sphere and is tangent to it at the point (a,0,0)—which now has a negative x-coordinate. The curve $a^2 = b$ and $a < 0$ has an "∞" shape moulded over the sphere. Since the curve crosses itself, it is not smooth. $\square$

2. Problem 3.1.2

(a) For what value of c is the set X_c of the equation $x^2 + y^3 = c$ a smooth curve?

Solution. If we define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = x^2 + y^3 - c$, so that X_c is the level set $F = 0$, then X_c is a smooth curve as long as DF is a surjective map, i.e., $[DF(x, y)] \neq 0$ in this case. We compute

$$[DF(x, y)] = \begin{bmatrix} 2x & 3y^2 \end{bmatrix}.$$

We can see that $[DF(x, y)] = 0$ if and only if $(x, y) = (0, 0)$, in which case $c = 0$. So X_c is a smooth curve for all non-zero values of c. □

(b) Give the equation of the tangent line at a point (u, v) of such a curve X_c.

Solution. The tangent space to the curve at the point $p = (u, v)$ is given by

$$T_p X_c = \ker[DF(p)] = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{bmatrix} 2u & 3v^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| 2ux + 3v^2 y = 0 \right\}.$$

Recall, however, that the tangent space is centered at the origin, so we need to shift it to (u, v) to get the tangent line. Hence the equation for the tangent line is

$$2u(x - u) + 3v^2(y - v) = 0.$$

□

(c) Sketch this curve for a representative sample of values of c

Solution. Most people got full credit for this. What is a representative sample anyways? You gotta love the Hubbards' rigorous language. I was looking for a curve with $c < 0$, one with $c = 0$ and one with $c > 0$. That seemed 'representative' enough. Most of you agreed. □