Problem Set #9c – Solutions

Mike Greene
Math 25

Problem 1.

Let $\mathcal{X}_l$ be the set of all line segments of length $l > 0$ in $\mathbb{R}^3$ with one endpoint on the x-axis and one endpoint on the unit sphere centered at the origin. Let $p = \begin{pmatrix} 1+l \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Claim 1.1. We can represent $\mathcal{X}_l$ as a the subset of $\mathbb{R}^4$ satisfying a set of equations.

Let's treat $\mathcal{X}$ as a subset of $\mathbb{R}^4$ in the following manner: Let $\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ correspond to the segment with one endpoint at $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the other at $\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$. Note that this representation is injective (i.e. no two distinct segments have the same representation as an element of $\mathbb{R}^4$). Note also, that by simple consideration of degrees of freedom, every segment in $\mathcal{X}_l$ can be represented in this manner.

We see, however, that not all points in $\mathbb{R}^4$ correspond to segments in $\mathcal{X}_l$ in this manner, and we wish find equations defining the subset of $\mathbb{R}^4$ that do. Our representation has already taken into account that any segment in $\mathcal{X}_l$ must have one endpoint on the x-axis, but we must now add the constraints that the segment have one endpoint on the unit sphere (i.e. that $x^2 + y^2 + z^2 = 1$) and have length l (i.e. that $(x-t)^2 + y^2 + z^2 = l^2$). We thus obtain that

$$\mathcal{X}_2 \cong \left\{ \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4 \mid \begin{pmatrix} x^2 + y^2 + z^2 - 1 \\ (x-t)^2 + y^2 + z^2 - l^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

□

Claim 1.2. Near the point p, the set $\mathcal{X}_2$ is a manifold.

We rewrite our algebraic constraints from the previous problem: define $f : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$f : \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x^2 + y^2 + z^2 - 1 \\ (x-t)^2 + y^2 + z^2 - l^2 \end{pmatrix}$$

We then see that

$$\mathcal{X}_2 \cong \left\{ \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4 \mid f\left(\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

We next caclulate that

$$Df\left(\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2(t-x) & 2(x-t) & 2y & 2z \end{pmatrix}$$

and thus

$$Df(p) = \left(\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2l & -2l & 0 & 0 \end{pmatrix}$$

which has rank 2 (due to the fact that the rows are linearly independent). We recall that if a set is given as the null space of a (continuous) map, then the set is a manifold around any point at which the derivative has maximal rank. 2 is the maximum possible rank for a map $\mathbb{R}^4 \to \mathbb{R}^2$, and thus $\mathcal{X}_l$ is a manifold around p.

To find the equation for tangent space to $\mathcal{X}_l$ at p, we simply take the kernel of $Df(p)$. A point is in the kernel of $Df(p)$ iff

$$\begin{aligned} \begin{pmatrix} 0 & 0 \end{pmatrix} &= Df(p)(\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}) \\ &= \begin{bmatrix} 2l & -2l & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 2lt - 2lx & 2x \end{pmatrix} \\ &= \begin{pmatrix} t - x & x \end{pmatrix} \\ &= \begin{pmatrix} t & x \end{pmatrix} \end{aligned}$$

Thus, $T_p\mathcal{X}_l = \{ \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \mid t = x = 0\}$

□

Claim 1.3. If $l \neq 1$, then $\mathcal{X}_l$ is a manifold.

We recall (from the previous part of this problem) that $\mathcal{X}_l$ is a manifold around every point at which the derivative has maximal rank. We will begin by row reducing the derivative obtained above:

$$\left(\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2(t-x) & 2(x-t) & 2y & 2z \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & x & y & z \\ t-x & -t & 0 & 0 \end{pmatrix}$$

If the two rows of this matrix were not linearly independent (i.e. this matrix had rank less than 2), we would have α, β, not both zero, such that $\alpha \cdot \begin{pmatrix} 0 & x & y & z \end{pmatrix} + \beta \cdot \begin{pmatrix} t-x & -t & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$ and thus

$$\begin{pmatrix} \beta(t-x) & \alpha x - \beta t & \alpha y & \alpha z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

We must prove that this is not the case, so we will assume that it is true and try to derive a contradiction with the constraint that $l \neq 1$. This is sort of ugly, but here goes: let $q := \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$

There are five cases:

1. **($x = t = 0$).** We recall that if $q \in \mathcal{X}_l$, then the condition that the segment have length l implies that

$$\begin{aligned} (x-t)^2 + y^2 + z + 2 - l^2 &= 0 \\ (x-0)^2 + y^2 + z^2 &= l^2 \end{aligned}$$

   and the condition that one end of the segment must lie on the unit sphere centered at the origin tells us that the left hand side, $x^2 + y^2 + z^2$, equals 1. Which would imply $l = 1$ which is a contradiction.

2. **($x = t \neq 0$ and $\alpha = 0$).** If $\alpha = 0$, then $\beta \neq 0$ and Our linear combination, (1), becomes

$$\begin{pmatrix} 0 & -\beta t & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

   which is a contradiction, as both β and t must be nonzero.

3. **($x = t \neq 0$ and $\alpha \neq 0$).** We recall that if $q \in \mathcal{X}_l$, then the condition that the segment have length l implies that

$$\begin{aligned} (x-t)^2 + y^2 + z + 2 - l^2 &= 0 \\ y^2 + z^2 &= l^2 > 0 \end{aligned}$$

   and thus at least one of y and z is nonzero. Our linear combination, (1), becomes:

$$\begin{pmatrix} 0 & (\alpha - \beta)t & \alpha y & \alpha z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

   either the third or fourth component of which is impossible as $\alpha \neq 0$ and at least one of y and z is nonzero.

4. **($x \neq t$ and $\alpha = \beta$).** If $\alpha = \beta$, then both must be nonzero, and the first component of (1),

$$\beta(t - x) = 0$$

   is impossible as both β and $t - x$ must be nonzero.

5. **($x \neq t$ and $\alpha \neq \beta$).** We recallconsider the difference of the first two components of (1):

$$\begin{aligned} 0 - 0 &= (\beta(t-x)) - (\alpha x - \beta t) \\ &= (\beta - \alpha)x \end{aligned}$$

   We conclude $x = 0$ therefore $t \neq 0$ by the assumption that $x \neq t$. Furthermore, from $x^2 + y^2 + z^2 = 1$, we conclude that at least one of y and z is nonzero. (1) becomes:

$$\begin{pmatrix} \beta t & -\beta t & \alpha y & \alpha z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

   We now have a contradiction: either $\beta \neq 0$ and then the first two components of the left hand side are nonzero, or $\alpha \neq 0$ and at least one of the third and fourth components of the left hand side is nonzero.

□

Problem 2.

Claim 2.1. Let $\mathcal{X}$ be the set of all line segments in $\mathbb{R}^3$ with length 2 and with endpoints on the unit spheres centered at $(1,0,0)$ and $(-1,0,0)$. Then $\mathcal{X}$ is a manifold except where the x-coordinate of one of the endpoints is ± 2.

We will begin by representing elements of $\mathcal{X}$ as elements of $\mathbb{R}^6$. Let $p \in \mathcal{X}$. We associate

$$p \leftrightsquigarrow \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

where x_1, y_1, and z_1 are the coordinates of the first endpoint, and x_2, y_2, and z_2 are the coordinates of the second endpoint. We then define $f : \mathbb{R}^6 \to \mathbb{R}^3$ by:

$$f\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} (x_1+1)^2 + y_1^2 + z_1^2 - 1 \\ (x_2-1)^2 + y_2^2 + z_2^2 - 1 \\ (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 - 4 \end{pmatrix}$$

We note that $\mathcal{X} \cong \ker(f)$: the first component being zero is equivalent to the first endpoint lying on the first sphere, the second component being zero is equivalent to the second endpoint lying on the second sphere, and the third component being zero is equivalent to the length of the rod being 2.

We now consider Df in order to determine the points near which $\mathcal{X}$ is a manifold:

$$Df\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = \begin{bmatrix} 2x_1+2 & 2y_1 & 2z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2x_2-2 & 2y_2 & 2z_2 \\ 2x_1-2x_2 & 2y_1-2y_2 & 2z_1-2z_2 & 2x_2-2x_1 & 2y_2-2y_1 & 2z_2-2z_1 \end{bmatrix}$$

We row-reduce slightly to obtain:

$$\begin{bmatrix} x_1+1 & y_1 & z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2-1 & y_2 & z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 & x_2-x_1 & y_2-y_1 & z_2-z_1 \end{bmatrix}$$

which we will call M. We know that $\mathcal{X}$ is a manifold around all points at which M is maximal rank. We will thus take an arbitrary linear combination of the three rows, and find conditions under which it can be zero. (We will first change the rows into columns so that they can fit on the page.)

$$M^T \cdot \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} = \alpha \begin{bmatrix} x_1+1 \\ y_1 \\ z_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_2-1 \\ y_2 \\ z_2 \end{bmatrix} + \gamma \begin{bmatrix} x_1-x_2 \\ y_1-y_2 \\ z_1-z_2 \\ x_2-x_1 \\ y_2-y_1 \\ z_2-z_1 \end{bmatrix} = \begin{bmatrix} \alpha(x_1+1)+\gamma(x_1-x_2) \\ \alpha y_1+\gamma(y_1-y_2) \\ \alpha z_1+\gamma(z_1-z_2) \\ \beta(x_2-1)+\gamma(x_2-x_1) \\ \beta y_2+\gamma(y_2-y_1) \\ \beta z_2+\gamma(z_2-z_1) \end{bmatrix} \tag{2}$$

We will now check that our conditions (that $x_1 \neq -2$ and $x_2 \neq 2$) are necessary and sufficient.

Necessity. If $x_1 = -2$, then we know $y_1 = z_1 = x_2 = y_2 = z_2 = 0$ and

$$\tilde{Df}\left(\begin{pmatrix}-2\\0\\0\\0\\0\\0\end{pmatrix}\right) = \begin{bmatrix}-2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -4 & 0 & 0 & 4 & 0 & 0\end{bmatrix}$$

which is clearly not maximal rank. Similarly, if $x_2 = 2$, then we have $x_1 = y_1 = z_1 = y_2 = z_2 = 0$ and thus

$$Df\left(\begin{pmatrix}0\\0\\0\\2\\0\\0\end{pmatrix}\right) = \begin{bmatrix}0 & 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ -4 & 0 & 0 & 4 & 0 & 0\end{bmatrix}$$

which is not invertible either.

Sufficiency. Suppose that our linear combination, (2), is zero. We will attempt to derive a contradiction with the fact that $x_1 \neq -2$ and $x_2 \neq 2$.

We first note that $\gamma \neq 0$, for suppose it was zero. Then we would have

$$\alpha \begin{bmatrix}x_1+1\\ y_1\\ z_1\\ 0\\ 0\\ 0\end{bmatrix} + \beta \begin{bmatrix}0\\0\\0\\ x_2-1\\ y_2\\ z_2\end{bmatrix} = \begin{bmatrix}(1+x_1)\alpha\\ y_1\alpha\\ z_1\alpha\\ (x_2-1)\beta\\ y_2\beta\\ z_2\beta\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix}$$

and, becuase at least one of α and β must be nonzero, we have either $y_1 = z_1 = 0$, which implies that $x_1 = -2$, or $y_2 = z_2 = 0$, which implies that $x_2 = 2$. Assuming $\gamma = 0$, thus, gives us a contradiction.

Knowing that $\gamma \neq 0$ we can divide our linear combination, (2), by $-\gamma$. Taking $\tilde{\alpha} := -\frac{\alpha}{\gamma}$ and $\tilde{\beta} := -\frac{\beta}{\gamma}$ we obtain:

$$\tilde{\alpha} \begin{bmatrix}x_1+1\\ y_1\\ z_1\\ 0\\ 0\\ 0\end{bmatrix} + \tilde{\beta} \begin{bmatrix}0\\0\\0\\ x_2-1\\ y_2\\ z_2\end{bmatrix} - \begin{bmatrix}x_1-x_2\\ y_1-y_2\\ z_1-z_2\\ x_2-x_1\\ y_2-y_1\\ z_2-z_1\end{bmatrix} = \begin{bmatrix}\tilde{\alpha}(x_1+1)-(x_1-x_2)\\ \tilde{\alpha}y_1-(y_1-y_2)\\ \tilde{\alpha}z_1-(z_1-z_2)\\ \tilde{\beta}(x_2-1)-(x_2-x_1)\\ \tilde{\beta}y_2-(y_2-y_1)\\ \tilde{\beta}z_2-(z_2-z_1)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix} \quad (3)$$

The first three rows of which give us:

$$\frac{x_1-x_2}{x_1+1} = \frac{y_1-y_2}{y_1} = \frac{z_1-z_2}{z_1} = \tilde{\alpha}$$

and thus we can derive a nice result from our condition that the length of the segment equal 2:

$$\begin{aligned} 4 &= (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2) \\ &= \tilde{\alpha}^2((x_1+1)^2 + y_1^2 + z_1) \\ &= \tilde{\alpha}^2 \end{aligned}$$

and thus $\tilde{\alpha} = \pm 2$. We can similarly use the last three rows of (3) to obtain

$$\frac{x_2 - x_1}{x_2 + 1} = \frac{y_2 - y1}{y_2} = \frac{z_2 - z_1}{z_2} = \tilde{\beta}$$

and thus $\tilde{\beta} = \pm 2$. There are now three cases:

1. ($\alpha = 2$). In this case, the first row of (3) gives us $x_1 + x_2 = -2$. Given that $x_1 \in [-2, 0]$ and $x_2 \in [0, 2]$, we see that $x_1 = -2$ and $x_2 = 0$. $\Rightarrow\Leftarrow$.
2. ($\beta = 2$). In this case, the fourth row of (3) give us $x_1 + x_2 = 2$ which implies $x_2 = 2$. $\Rightarrow\Leftarrow$.
3. ($\alpha = -2$ and $\beta = 2$). In this case, we can simultaneously solve the first and fourth row of (3) to obtain $x_1 = 0$ and $x_2 = 2$. $\Rightarrow\Leftarrow$

$\square$