

**MATH 25A – EXAM #1
SOLUTIONS**

- (1) Let $\{a_i\}$ be a sequence in \mathbb{R}^n that converges to a . Prove that if $v \in \mathbb{R}^n$ then the sequence of dot products $\{v \cdot a_i\}$ converges to $v \cdot a$. (Give a complete ε - N proof for full credit.)

Proof. Let $\varepsilon > 0$. If $v = 0$, then $\{a_i \cdot v\}$ is the zero sequence, which converges to $a \cdot v = 0$. If $v \neq 0$, then because $a_i \rightarrow a$, there exists N such that $|a_i - a| < \varepsilon/|v|$ for $i > N$. Now

$$|a_i \cdot v - a \cdot v| \leq |a_i - a| \cdot |v| < \varepsilon$$

for $i > N$. □

- (2) In each case below, either prove that the subset W of the vector space V is a subspace, or explain why not.

- (a) Let $L_1, L_2 : V \rightarrow U$ be two linear maps, and

$$W = \{v \in V \mid L_1(v) = L_2(v)\}.$$

Proof. W is nonempty because $L_1(0) = L_2(0)$, hence $0 \in W$. W is also closed under addition and scalar multiplication: for any $w_1, w_2, w \in W$ and $\alpha \in \mathbb{R}$

$$L_1(w_1 + w_2) = L_1(w_1) + L_1(w_2) = L_2(w_1) + L_2(w_2) = L_1(w_1 + w_2)$$

$$L_1(\alpha w) = \alpha L_1(w) = \alpha L_2(w) = L_1(\alpha w)$$

Hence both $w_1 + w_2 \in W$ and $\alpha w \in W$. □

- (b) Let $V = \text{Mat}(n, n)$, and $U \subset \mathbb{R}^n$ a subspace. Define

$$W = \{A \in V \mid Au \in U \text{ for all } u \in U\}.$$

Proof. W is nonempty because $0 \cdot u = 0 \in U$ for any $u \in U$, so the zero matrix lies in W . To check that W is closed under addition and scalar multiplication, let $A, B \in W$, $\alpha \in \mathbb{R}$:

$$(A + B)(u) = Au + Bu \in U \text{ for any } u \in U$$

$$(\alpha A)(u) = \alpha(Au) \in U \text{ for any } u \in U$$

□

- (3) Answer each question below “yes” or “no”. No proof is needed.

- (a) $\mathbb{Z} \subset \mathbb{R}$ (the set of integers) is closed.

Yes.

- (b) $\mathbb{Q} \subset \mathbb{R}$ (the set of rationals) is closed.

No. There exists a sequence of rationals $\{3, 3.1, 3.14, \dots\}$ whose limit is not rational.

- (c) There exists a continuous surjective map $f : [0, 1] \rightarrow \mathbb{R}$.
No. $[0, 1]$ is compact, its image must also be compact.
- (d) There exists a continuous surjective map $f : \mathbb{R} \rightarrow [0, 1]$.
Yes. $|\sin(x)|$, for example.
- (e) If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is any continuous function then the level set $f^{-1}(c)$ is closed for any $c \in \mathbb{R}$.
Yes. The set $\{c\}$ is closed, its inverse image $f^{-1}(c)$ is also closed.
- (f) If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is any continuous function then the level set $f^{-1}(c)$ is compact for any $c \in \mathbb{R}$.
No. For example, let f map \mathbb{R}^n to one point c .
- (4) Let $C \in \mathbb{R}^n$ be a closed nonempty set. Define the distance from a point $p \in \mathbb{R}^n$ to C by

$$d(p, C) = \inf_{c \in C} |p - c|.$$

- (a) Prove that this infimum exists and $d(p, C) \geq 0$ for any p .
Proof. The set $\{|p - c| \mid c \in C\}$ is nonempty (because C is nonempty) and bounded from below by 0, hence it has a greatest lower bound. \square
- (b) Prove that $d(p, C) = 0$ if and only if $p \in C$.
Proof. If $p \in C$, then $0 = |p - p|$ lies in the set defined above, hence 0 is the greatest lower bound. Conversely, if $d(p, C) = 0$ then there exists a sequence $a_i \in C$ such that $|p - a_i| < 1/i$. Hence $a_i \rightarrow p$ as $i \rightarrow \infty$. Since C is closed, the limit point p must lie in C . \square
 Many people wrote that because $d(p, C) = 0$, there must exist a $c \in C$ such that $|p - c| = 0$. This is true in our situation because C is closed, but not in general. For example, if $S = (0, 1) \subset \mathbb{R}$ and $p = 1$, then $d(p, S) = 0$, but $p \notin S$. I did not give any points for such a proof unless you showed that such c can always be found.
- (c) Prove the “triangle inequality”:

$$d(q, C) \geq d(p, C) - |p - q|$$

for any $p, q \in \mathbb{R}^n$. (Hint: the left hand side is the greatest lower bound of a set. Prove that the right hand side is a lower bound for the same set.)

Proof. From the usual triangle inequality and $|p - c| \geq d(p, C)$, we get

$$|q - c| \geq |p - c| - |p - q| \geq d(p, C) - |p - q|$$

This inequality is true for all c , hence the right hand side (which does not depend on c) is a lower bound for all values on the left hand side as $c \in C$. By the greatest lower bound property,

$$d(q, C) \geq d(p, C) - |p - q|$$

\square

(d) Prove that $d(\cdot, C)$ as a function $\mathbb{R}^n \rightarrow \mathbb{R}$ is continuous.

Proof. From the previous problem we get:

$$d(p, C) - d(q, C) \leq |p - q|$$

$$d(q, C) - d(p, C) \leq |q - p|$$

for all $p, q \in \mathbb{R}^n$. Combining these two:

$$|d(p, C) - d(q, C)| \leq |p - q|$$

Now let $\varepsilon > 0$. then

$$|d(p, C) - d(q, C)| < \varepsilon$$

whenever $|p - q| < \varepsilon$. □