MATH 25A – EXAM #2
NOVEMBER 14, 2001

(1) Let $f : Mat(n,n) \to Mat(n,n)$ be the map $f(A) = A \cdot A$. Prove that the derivative of f at A is:

$$Df(A) : H \mapsto AH + HA$$

(You may identify $Mat(n,n)$ with $\mathbb{R}^{n^2}$. Then both f and the derivative $Df(A)$ are maps $\mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$.)

Proof. Recall that a linear map $L : Mat(n,n) \to Mat(n,n)$ is a derivative of f at A if

$$\lim_{H \to 0} \frac{|f(A+H) - f(A) - L(H)|}{|H|} = 0.$$

The map $H \mapsto AH + HA$ is linear (for a fixed A), so we compute the limit:

$$\lim_{H\to 0} \frac{|(A+H)(A+H) - AA - AH - HA|}{|H|} = \lim_{H\to 0} \frac{|HH|}{|H|} \le \lim_{H\to 0} \frac{|H|^2}{|H|} = \lim_{H\to 0} |H| = 0.$$

□

(2) Answer each question "true" or "false". No proof is needed.

(a) If f is a composition of differentiable maps $f : \mathbb{R}^2 \to \mathbb{R} \to \mathbb{R}^2$ then the Jacobian of f can not be invertible at any point.
True. By the chain rule, the derivative of f at a also has to be a composition $\mathbb{R}^2 \to \mathbb{R} \to \mathbb{R}^2$. The kernel of $Df(a)$ is nonzero because it contains the kernel of $\mathbb{R}^2 \to \mathbb{R}$, which is nonzero.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that both partial derivatives of f exist and are zero at every point of $\mathbb{R}^2$. Then f must be differentiable in $\mathbb{R}^2$.
True. The partial derivatives are continuous, hence f is differentiable.

(c) Let $L : V \to W$ be a linear map with $Ker(L) = 0$. Then $\{L(v_1), \ldots, L(v_n)\}$ is a basis of W for any basis $\{v_1, \ldots, v_n\}$ of V.
False. $\{L(v_1), \ldots, L(v_n)\}$ is a basis of the image of L, but it may not span W.

(d) Let $L : V \to W$ be a linear map. Then $\{L(v_1), \ldots, L(v_n)\}$ spans $Img(L)$ for any $\{v_1, \ldots, v_n\}$ that spans V.
True.

(e) If $f \circ g$ is differentiable then both f and g must be differentiable.
False. For example, g may be constant, then $f \circ g$ is constant, hence differentiable, for any f.

(3) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ be two linear maps satisfying the following conditions:

f is injective;
g is surjective;
the composition $g \circ f$ is zero.

Prove that then $m \geq n + p$.

Proof. Apply the dimension formula for both g and f:

$$m = \dim Ker(g) + \dim Img(g) = \dim Ker(g) + p$$
$$n = \dim Ker(f) + \dim Img(f) = \dim Img(f).$$

Because the composition $g \circ f$ is zero, $Img(f) \subset Ker(g)$, hence $\dim Img(f) \leq \dim Ker(g)$. Thus we get

$$m = \dim Ker(g) + p \geq \dim Img(f) + p = n + p.$$

□

(4) Let P_d be the set of polynomials of degree d or less for some $d > 0$. Prove that for any $q(x) \in P_d$ there exists a $p(x) \in P_d$ such that

$$p(1) + xp'(1) + x^2p''(1) + \ldots x^d p^{(d)}(1) = q(x).$$

Proof. We have to show that the linear map

$$L : P_d \to P_d$$
$$L : p(x) \mapsto p(1) + xp'(1) + x^2p''(1) + \ldots x^d p^{(d)}(1)$$

is surjective. This is equivalent to showing that the kernel of L is zero. Suppose $p(x) \neq 0$ lies in the kernel. Write

$$p(x) = a_k x^k + \ldots + a_0,$$

where $a_k \neq 0$, $0 \leq k \leq d$. Since $p(x)$ lies in the kernel of L, $L(p)$ is a polynomial, all of whose coefficients are zero. Consider the coefficient of L(p) in front of x^k: this is $p^{(k)}(1) = a_k k!$. For this to be zero, $a_k = 0$, and we get a contradiction. □

(5) Let C be the cardioid given as the image of the map

$$\phi : [-\pi, \pi] \to \mathbb{R}^2$$
$$t \mapsto \begin{pmatrix} (1 + \cos(t)) \cos(t) \\ (1 + \cos(t)) \sin(t) \end{pmatrix}$$

One can check (you don't have to do this) that for any $t \in (-\pi, \pi)$ the Jacobian of ϕ at t is a nonzero vector tangent to the Cardioid, hence we can use it to draw a tangent line to the Cardioid through the point $\phi(t)$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any differentiable function. For $x \in \mathbb{R}^2$, define the gradient of f at x by

$$grad(f)(x) = \begin{bmatrix} D_1 f(x) \\ D_2 f(x) \end{bmatrix}$$

In other words, $grad(f)(x)$ is the transpose of the Jacobian of f.

Prove that there exists a point $c \in C$, different from the cusp $\phi(-\pi) = \phi(\pi)$, such that either $grad(f)(c) = 0$ or $grad(f)(c)$ is perpendicular to the tangent line to C at c.

Proof. Apply the one-variable mean value theorem to the composition $f \circ \phi$: there exists a $t_0 \in (-\pi, \pi)$ such that

$$D(f \circ \phi)(t_0)(\pi - (-\pi)) = f \circ \phi(\pi) - f \circ \phi(-\pi).$$

The right hand side is zero because $\phi(\pi) = \phi(-\pi)$. To the left hand side we can apply the chain rule. Writing it in terms of products of Jacobian matrices, we get:

$$Jf(\phi(t_0)) \cdot J\phi(t_0) \cdot 2\pi = 0.$$

Here the product of the two Jacobians is a product of matrices. As a dot product:

$$grad(f)(\phi(t_0)) \cdot J\phi(t_0) = 0.$$

Since $J\phi(t_0) \neq 0$, we see that grad(f) at $c = \phi(t_0)$ is perpendicular to the tangent line to C at c. $\square$