MATH 25A – PROBLEM SET #1
DUE FRIDAY SEPTEMBER 27

1. Part A

There are two parts in this homework. Please turn them in as separate sets so that they can be graded separately.

The first part of the homework contains problems from set theory. If you are not familiar with notions such as "is an element of", "subset", "intersection", etc, please read Section 0.3 in the textbook. Here is some more terminology concerning maps (= mappings = functions) between sets:

Definition. Let $f : S \to T$ be a map of sets.

- If $A \subset S$, then $f(A) = \{f(s) | s \in A\} \subset T$ is called the *image* of A.
- If $B \subset T$, then $f^{-1}(B) = \{s \in S | f(s) \in B\} \subset S$ is called the *inverse image* of B. When $B = \{t\}$ consists of one element, the inverse image is usually denoted by $f^{-1}(\{t\}) = f^{-1}(t)$.

One must be careful not to confuse f^{-1} with the inverse map (if such exists). In particular, $f^{-1}(t)$ is a subset of S, not an element of S.

Definition. A map $f : S \to T$ is called

- *injective* (or one-to-one) if $f(s_1) = f(s_2)$ implies that $s_1 = s_2$ for any $s_1, s_2 \in S$.
- *surjective* (or onto) if $f(S) = T$.
- *bijective* if it is both injective and surjective.

Most problems in this section are relatively simple. The emphasis here is on rigorous proofs. When writing a proof, one should keep in mind that a good proof is:

- Correct – ideally, every statement should follow from axioms or from what has been proved before.
- Concise – a proof should not contain anything that is not necessary. Nobody, including the grader of your homework, likes to read very long proofs.
- Readable – proofs are written for people, not computers. Don't be afraid to clarify in words what you are doing. For example, before embarking on a long computation, it is a good idea to explain what you are going to do and why you are going to do it.

Here is an example of a proof that is straight-forward in structure:

Theorem. *Let $f : S \to T$ be a map of sets. For any $A, B \subset S$, we have*

$$f(A \cup B) = f(A) \cup f(B).$$

Proof. To prove that the two sets are equal, we first show that $f(A \cup B) \subset f(A) \cup f(B)$ and then the converse $f(A) \cup f(B) \subset f(A \cup B)$.

Let $t \in f(A \cup B)$, hence $t = f(s)$ for some $s \in A \cup B$. Since $s \in A \cup B$, either $s \in A$ or $s \in B$. In the first case we have that $t = f(s) \in f(A)$, and in the second case $t = f(s) \in f(B)$. Thus, $t \in f(A) \cup f(B)$.

For the converse, let $t \in f(A) \cup f(B)$, that means, either $t \in f(A)$ or $t \in f(B)$. In the first case, $t = f(s)$ for some $s \in A \subset A \cup B$, hence $t \in f(A \cup B)$. The second case is similar. $\square$

(1) Let $f : S \to T$ be a map of sets, $A, B \subset S$ and $C, D \subset T$.
   (a) Prove that $f(A \cap B) \subset f(A) \cap f(B)$.
   (b) Prove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

(2) Let $A, B \subset S$. Prove that $A \subset B$ if and only if $S - B \subset S - A$. (Note that this is an if and only if statement. So you have to prove that the first statement implies the second, and conversely, that the second implies the first.)

(3) Let $f : S \to T$ be a map of sets. Prove that the following are equivalent:
   (a) f is injective.
   (b) $f^{-1}(t)$ contains at most one element for any $t \in T$.
   (c) $f(A \cap B) = f(A) \cap f(B)$ for any $A, B \subset S$.
   (Hint: it suffices to prove that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.)

(4) Use induction to prove the following:
   (a) $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$.
   (b) $\sum_{i=1}^{n} 3^{2i-1} = 3(9^n - 1)/8$.

Definition. A relation $\sim$ between elements of a set S is called an *equivalence* relation if for any $s, t, u \in S$ it satisfies:

- (reflexivity) $s \sim s$,
- (symmetry) if $s \sim t$ then $t \sim s$,
- (transitivity) if $s \sim t$ and $t \sim u$ then $s \sim u$.

(5) On the set $\mathbb{Z}$ define $m \sim n$ if and only if $m - n$ is divisible by 12. Show that this is an equivalence relation.

Definition. For any $a \in S$ we define the equivalence class of a as the set of all elements of S equivalent to a:

$$\overline{a} = \{s \in S | a \sim s\}.$$

An element $s \in \overline{a}$ is called a representative of the equivalence class.

(6) Prove that if $\overline{a}, \overline{b}$ are two equivalence classes, then either $\overline{a} = \overline{b}$ or $\overline{a} \cap \overline{b} = \emptyset$.

Definition. The quotient of S by the equivalence relation $\sim$ is the set of all equivalence classes:

$$S/\sim = \{\overline{a} | a \in S\}.$$

(7) Let S be the set of all pairs of integers (a, b) such that $b \neq 0$. Define $(a, b) \sim (c, d)$ if and only if $ad = bc$.
   (a) Prove that this is an equivalence relation.
   (b) Does the quotient $S/\sim$ look familiar?

(8) Let $f : S \to T$ be a map of sets. Define a relation in S by $s_1 \sim s_2$ iff $f(s_1) = f(s_2)$.
   (a) Prove that this is an equivalence relation.
   (b) find an injective map

$$\overline{f} : S/\sim \to T$$

   (Here you have to construct the map and prove that it is injective.)

2. Part B

(1) Let $\{a_i\}$ and $\{b_i\}$ be two sequences in $\mathbb{R}$.
   (a) Prove that if $\{a_i\}$ converges to a and $\{b_i\}$ converges to b then the sequence $\{a_i + b_i\}$ converges to $a + b$.
   (b) Prove that if $\{a_i\}$ converges to a and the sequence $\{a_i - b_i\}$ converges to 0, then $\{b_i\}$ also converges to a.

(2) Let $\{a_i\}$ be a sequence in $\mathbb{R}$. Define a new sequence

$$b_i = \frac{a_1 + \ldots + a_i}{i}.$$

   (a) Prove that if $\{a_i\}$ converges to a then $\{b_i\}$ also converges to a. (Hint: this is slightly tricky. The easiest way to estimate the distance between b_i and a is to divide the sum in the numerator into two parts such that in the second part all a_j's are "small".)
   (b) find a sequence $\{a_i\}$ that does not converge, such that $\{b_i\}$ converges.

Consider the two properties of real numbers:

C1. Every nonempty subset $X \subset \mathbb{R}$ which is bounded from above has a least upper bound.

C2. Every non-decreasing bounded from above sequence in $\mathbb{R}$ has a limit.

(3) In class we proved that $C1$ implies $C2$. Prove the converse: $C2$ implies $C1$. (Hint: you have to take an arbitrary nonempty set $X \subset \mathbb{R}$ which is bounded from above, and assuming $C2$, prove that it has a least upper bound. The way to do it is to construct a monotone sequence of real numbers that converges to the sought after least upper bound. To construct such a sequence, you may want to look at the proof of the least upper bound property in the textbook.)

Definition. A sequence $\{a_i\}$ is called a *Cauchy sequence* if for any $\epsilon > 0$ there exists N such that

$$|a_i - a_j| < \epsilon$$

whenever $i, j > N$.

(4) Prove that if a sequence converges then it is a Cauchy sequence.

(5) Prove that a Cauchy sequence is bounded (from above and below).

(6) Let $\{a_i\}$ be a Cauchy sequence. Define a new sequence

$$b_i = \sup\{a_i, a_{i+1}, a_{i+2}, \ldots\}.$$

   (a) Prove that the sequence $\{b_i\}$ converges to some number b.
   (b) Prove that the sequence $\{a_i\}$ also converges to the same number b. (Hint: problem (1)(b) may be helpful.)

(7) Let $C3$ be the property that every Cauchy sequence in $\mathbb{R}$ converges. Prove that $C3$ is equivalent to $C1$ (or to $C2$). (Hint: everything has been proved in previous problems, it is only a matter of putting it all together.)