MATH 25A – PROBLEM SET #2
DUE FRIDAY OCTOBER 4

1. Part A

In problems 3, 4 and 5 you are asked to check the vector space axioms. Convince yourself that all the axioms hold, but write down only the proofs for axioms 1 (existence of zero), 2 (existence of inverse) and one more of your choice.

(1) Let $W_1, W_2 \subset V$ be subspaces of a vector space V.
  (a) Prove that $W_1 \cap W_2$ is also a subspace of V.
  (b) Prove that $W_1 \cup W_2$ is a subspace of V if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.

(2) Let $T : V \to W$ be a linear map. Recall that $T(V)$ is called the *image* of T. The *kernel* of T is $Ker(T) = T^{-1}(0)$.
  (a) Prove that $T(V) \subset W$ is a subspace.
  (b) Prove that $Ker(T) \subset V$ is a subspace.

(3) Let $Lin(V, W)$ be the set of linear maps from V to W. If $W = \mathbb{R}$ then $V^* = Lin(V, \mathbb{R})$ is called the *dual* of V.
  (a) Define addition and multiplication with scalars in $Lin(V, W)$ and prove that $Lin(V, W)$ is a vector space.
  (b) Recall that linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ are matrices of certain dimension. Describe the matrices in $(\mathbb{R}^n)^*$.

(4) Let V and W be vector spaces. Define addition and scalar multiplication in $V \times W$ and show that these make $V \times W$ into a vector space.

(5) Let $W \subset V$ be a subspace. Define a relation $\sim$ in V by: $v_1 \sim v_2$ if $v_1 - v_2 \in W$.
  (a) Prove that $\sim$ is an equivalence relation.
  (b) Let $V/W = V/\sim$ be the quotient of V. Prove that V/W is a vector space, with addition and scalar multiplication defined using representatives. (More precisely, you have to prove that addition and multiplication are well defined, independent of which representatives we choose, and then check the vector space axioms. To avoid confusion between lines and arrows on top of vectors, denote the equivalence class of a vector $\vec{v}$ by $[\vec{v}]$. For example, addition is then defined by $[\vec{v}_1] + [\vec{v}_2] = [\vec{v}_1 + \vec{v}_2]$.)

(6) Define the transpose of a matrix $A = (a_{ij})$ by $A^t = (a_{ji})$.
  (a) Prove that $(A \cdot B)^t = B^t \cdot A^t$.
  (b) Prove that if A is invertible then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

2. Part B

Definition. A *metric space* (S, d) consists of a nonempty set S together with a function $d : S \times S \to \mathbb{R}$ that assigns to two points $p, q \in S$ the distance between them, satisfying the following properties for all $p, q, r \in S$:

- $d(p, q) \geq 0$.
- $d(p, q) = 0$ if and only if $p = q$.
- $d(p, q) = d(q, p)$.
- $d(p, q) \leq d(p, r) + d(r, q)$.

The function d is called a *metric* on S.

One can define notions such as convergence of sequences, Cauchy sequences, continuous functions, and so on, in a metric space in the same way as in $\mathbb{R}$ by replacing $|x-y|$ with $d(p,q)$.

(1) Prove that the following are metric spaces:
  (a) $S=\mathbb{R}^n$, $d_2(p,q)=\sqrt{(p_1-q_1)^2+\ldots+(p_n-q_n)^2}$. (The triangle inequality will be proved in class.)
  (b) $S=\mathbb{R}^n$, $d_1(p,q)=|p_1-q_1|+\ldots+|p_n-q_n|$.
  (c) $S=\mathbb{R}^n$, $d_\infty(p,q)=\max\{|p_1-q_1|,\ldots,|p_n-q_n|\}$.
  (d) $S=C([0,1])$, $d_1(f,g)=\int_0^1|f(x)-g(x)|dx$.
  (e) $S=C([0,1])$, $d_\infty(f,g)=\max_x|f(x)-g(x)|$.

  (Here $C([0,1])$ is the set of continuous functions $f:[0,1]\to\mathbb{R}$. We will prove later that for such functions integrals and maxima always exist.)

(2) Prove that if (S,d) is a metric space and $T\subset S$ a subset, then (T,d_T) is again a metric space, with d_T coming from the metric on S.

(3) For the metrics d_1,d_2,d_∞ on $\mathbb{R}^2$, draw the open unit ball around 0:

$$B_1(0)=\{p\in\mathbb{R}^2|d(p,0)<1\}.$$

(4) Two metrics d_1 and d_2 on a set S are said to be *equivalent* if for any point $p\in S$, every open ball $B^1_R(p)$ with respect to the first metric contains an open ball $B^2_r(p)$ with respect to the second metric, and vice versa, every open ball $B^2_R(p)$ with respect to the second metric contains an open ball $B^1_r(p)$ with respect to the first metric.
  (a) Prove that if d_1,d_2 are equivalent, then a sequence $\{a_i\}$ in S converges with respect to d_1 if and only if it converges with respect to d_2.
  (b) Prove that the metrics d_1,d_2,d_∞ on $\mathbb{R}^n$ are all equivalent.
  (c) Prove that the metrics d_1 and d_∞ on $C[0,1]$ are not equivalent by finding a sequence that converges with respect to d_1 but does not converge with respect to d_∞.

(5) A metric space is said to be *complete* if every Cauchy sequence converges. Prove that $(\mathbb{R}^n,d_2)$ is complete. (Hint: it may be easier to prove that $(\mathbb{R}^n,d_\infty)$ is complete. Does this imply the completeness of d_2?)

(6) Let (S,d) be a complete metric space and $f:S\to S$ a map. Then f is called a *contraction map* if there exists a constant $0\le C<1$ such that for all $p,q\in S$ we have

$$d(f(p),f(q))<Cd(p,q).$$

  In other words, a contraction map decreases distance between points by at least a constant factor.

  The goal of this exercise is to show that every contraction map has a unique fixed point $p=f(p)$.
  (a) Prove that a contraction map can have at most one fixed point.
  (b) Choose a point $p\in S$ and consider the sequence of iterates $(a_0=p,a_1=f(p),a_2=f(f(p)),\ldots)$. Prove that this is a Cauchy sequence, hence it converges. (Hint: find the distance $d(a_i,a_{i+1})$ in terms of $d(a_0,a_1)$, then estimate $d(a_i,a_j)$ for $j>i$.)
  (c) Prove that the limit of the sequence in the previous problem is a fixed point of f. (Hint: turn the following argument into a formal proof. We show that the distance between the limit point a and $f(a)$ is arbitrarily small (hence it is zero). For large i we have that a_i is very close to the limit a. By the contraction property, $f(a_i)$ is also close to $f(a)$. But $f(a_i)=a_{i+1}$ is close to the limit a, hence $f(a)$ is close to a.)