

MATH 25A – PROBLEM SET #3
DUE FRIDAY OCTOBER 11

1. PART A

- (1) Problem 1.5.5 in the textbook.
- (2) Problem 1.5.16 in the textbook.
- (3) Prove that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$ if and only if for any sequence $\{a_i\}$ in \mathbb{R}^n such that $\lim_{i \rightarrow \infty} a_i = a$, we have that $\lim_{i \rightarrow \infty} f(a_i) = f(a)$.
- (4) Prove that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$ and $g : \mathbb{R}^m \rightarrow \mathbb{R}^k$ is continuous at $f(a)$ then $g \circ f$ is continuous at a .
- (5) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \end{cases}$$

Here $x = \frac{p}{q}$ means that p and $q > 0$ are integers with no common divisors. Prove that f is continuous at all irrational points and f is not continuous at any rational point.

- (6) A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *convex* if for any $u, v \in [a, b]$ and any $0 \leq \lambda \leq 1$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v).$$

Prove that a convex function is continuous in the open interval (a, b) . Does it have to be continuous at the endpoints? (Hint: To prove continuity at a point $c \in (a, b)$, draw two lines in \mathbb{R}^2 : one through the points $(a, f(a))$ and $(c, f(c))$, and another line through $(b, f(b))$ and $(c, f(c))$. These lines divide the plane into four quadrants. The graph of f has to lie in two of the four quadrants.)

2. PART B

Definition. A *normed space* $(V, \|\cdot\|)$ consists of a vector space V together with a function $\|\cdot\| : V \rightarrow \mathbb{R}$ that assigns to a vector $v \in V$ its *norm* $\|v\|$, satisfying the following properties for all $v, w \in V$, $\alpha \in \mathbb{R}$:

- $\|v\| \geq 0$.
- $\|v\| = 0$ if and only if $v = 0$.
- $\|\alpha v\| = |\alpha| \|v\|$.
- $\|v + w\| \leq \|v\| + \|w\|$.

- (1) Prove that if $(V, \|\cdot\|)$ is a normed space, then

$$d(v, w) = \|v - w\|$$

is a metric on V .

- (2) Consider the five metrics defined in Homework #2. Show that each metric comes from a norm. Prove that these really are norms.
- (3) Define the norm of a matrix $A \in \text{Mat}(n, m)$ by

$$\|A\| = \sup_{0 \neq v \in \mathbb{R}^m} \frac{|Av|}{|v|}.$$

Prove that $\|A\|$ exists for any A and that this defines a norm on $\text{Mat}(n, m)$.

- (4) A subset $S \subset V$ of a vector space is called *convex* if for any $u, v \in S$ and for any $0 \leq \lambda \leq 1$, we have that $\lambda u + (1 - \lambda)v \in S$. If $\|\cdot\|$ is a norm on V , show that the open ball with radius $R > 0$

$$B_R = \{v \in V \mid \|v\| < R\}$$

is convex.

- (5) Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two norms on a vector space V . We say that $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ if there exists a constant $C > 0$ such that

$$\|v\|_\alpha \leq C\|v\|_\beta$$

for any $v \in V$. The two norms are said to be equivalent if $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ and $\|\cdot\|_\beta \leq \|\cdot\|_\alpha$.

- (a) Prove that $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ if and only if for some $C > 0$

$$B_1^\beta \subset B_C^\alpha.$$

Here B_R^i is the open ball of radius R with respect to the norm $\|\cdot\|_i$, $i = \alpha, \beta$.

- (b) Prove that two norms are equivalent if and only if they define equivalent metrics.
- (c) Prove that the three norms on \mathbb{R}^n defined in problem (2) are all equivalent. (You may quote results from previous homeworks.)
- (6) The goal of this exercise is to prove that all norms on \mathbb{R}^n are equivalent by showing that an arbitrary norm $\|\cdot\|_\alpha$ on \mathbb{R}^n is equivalent to the norm $\|\cdot\|_1$ defined in problem (2) (the “sum of absolute values” norm).

- (a) Let $M = \max_i \|\vec{e}_i\|_\alpha$. Using triangle inequality prove that for any $v \in \mathbb{R}^n$

$$\|v\|_\alpha \leq M\|v\|_1.$$

- (b) Prove that the function $\|\cdot\|_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous.

- (c) Let $S = \{v \in \mathbb{R}^n \mid \|v\|_1 = 1\}$ be the unit sphere with respect to the norm $\|\cdot\|_1$ (so it is actually the boundary of a “diamond”). We consider the norm $\|\cdot\|_\alpha$ restricted to S :

$$\|\cdot\|_\alpha : S \rightarrow \mathbb{R}.$$

We will prove in class that such a function (a continuous function on a compact set) achieves its minimum m at some point $u \in S$. Assuming this, prove that $m \neq 0$ and

$$\|v\|_1 \leq \frac{1}{m}\|v\|_\alpha$$

for any $v \in \mathbb{R}^n$.