MATH 25A – PROBLEM SET #4
DUE FRIDAY OCTOBER 25

1. Part A

(1) Problem 1.7.8 in the textbook.
(2) Problem 1.7.21 in the textbook. (You have to recall the definition of the determinant of a 2×2 and consider this as a map $det : \mathbb{R}^4 \to \mathbb{R}$.)
(3) Problem 1.9.2(a) in the textbook.
(4) Recall the polar coordinates in $\mathbb{R}^2$.
  (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Write the Jacobian matrix of f with respect to polar coordinates
$$[\frac{\partial f}{\partial r} \quad \frac{\partial f}{\partial \theta}]$$
  in terms of the usual partial derivatives of f. (Hint: compose f with the "change of variables" map
$$\mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$
  (b) Problem 1.8.6(a) in the textbook. You may assume that $f : \mathbb{R}^2 \to \mathbb{R}$.
(5) Let $f, g : \mathbb{R} \to \mathbb{R}^3$ be two differentiable functions. The cross product
$$f \times g : \mathbb{R} \to \mathbb{R}^3$$
is a polynomial in f and g, hence differentiable. Find the Jacobian matrix of $f \times g$. Write your answer in terms of cross products. (The definition of the cross product of two vectors is given on p.81–82 in the textbook.)

2. Part B

(1) A function $f : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous* of degree m if
$$f(kx) = k^m f(x)$$
for all $k \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Let f be a homogeneous function of degree m, and assume that f has continuous partial derivatives. Prove that f satisfies the equation
$$\sum_{i=1}^{n} x_i D_i f(x) = mf(x)$$
for all $x \in \mathbb{R}^n$. (Hint: interpret the left hand side as a directional derivative.)
(2) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *Lipschitz* if there exists $M > 0$ such that
$$|f(x) - f(y)| < M|x - y|$$
for all $x, y \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *locally Lipschitz* if for every $x \in \mathbb{R}^n$ there exists $M > 0$ and $\delta > 0$ such that
$$|f(x) - f(y)| < M|x - y|$$
for all $y \in \mathbb{R}^n$ satisfying $|x - y| < \delta$.
  (a) Prove that a locally Lipschitz function is continuous.
  (b) Prove that a differentiable function is locally Lipschitz.

(c) Show that the two implications above are strict: there exists a continuous function that is not locally Lipschitz, and there exists a locally Lipschitz function that is not differentiable. (Hint: You can find such functions $f : \mathbb{R} \to \mathbb{R}$.)

(3) Consider the generalization of the mean value theorem to differentiable maps $f : \mathbb{R}^n \to \mathbb{R}^m$: given $a, b \in \mathbb{R}^n$, there exists c on the line segment $[a, b]$ such that

$$f(b) - f(a) = Df(c)(b - a).$$

Show that this generalization is not true for $m > 1$. (Hint: consider a map $f : \mathbb{R} \to \mathbb{R}^m$, which can pictured as a particle moving in $\mathbb{R}^m$. Then $Df(c)(\vec{e}_1)$ is the velocity vector, tangent to the trajectory.) Is it true that for any differentiable map $f : \mathbb{R} \to \mathbb{R}^2$ and any two points $a, b \in \mathbb{R}$, there exists $c \in [a, b]$ and $\lambda \in \mathbb{R}$ such that

$$f(b) - f(a) = \lambda Df(c)(b - a)?$$