MATH 25A – PROBLEM SET #6
DUE FRIDAY NOVEMBER 8

1. Part A

(1) Find a basis for $Mat(n,m)$. If V is a vector space with basis $\{v_1,\dots,v_n\}$ and W is a vector space with basis $\{w_1,\dots,w_n\}$, find a basis for the space $Lin(V,W)$.
(2) Problem 2.4.12(a) in the textbook.
(3) Prove that $\{v_1,\dots,v_n\}$ is a basis of V if and only if for any vector space W and any vectors $w_1,\dots,w_n\in W$ there exists a unique linear map $L:V\to W$ such that $L(v_i)=w_i$ for $i=1,\dots,n$.
(4) Let V and W be subspaces of $\mathbb{R}^n$. Define $V+W=\{v+w|v\in V,w\in W\}$. It is easy to see that this is a subspace of $\mathbb{R}^n$. Prove that

$$\dim V+W=\dim V+\dim W-\dim V\cap W.$$

2. Part B

(1) Let V be a vector space with basis $\{v_1,\dots,v_n\}$ and W a vector space with basis $\{w_1,\dots,w_m\}$. Also let $L:V\to W$ be a linear map with matrix A with respect to the two bases.
   (a) Prove that $\{w_1,\dots,w_{i-1},\frac{1}{\alpha}w_i,w_{i+1},\dots,w_m\}$ for any $1\le i\le m$ and $\alpha\neq 0$ is again a basis of W. Find the matrix of L with respect to this new basis in terms of A.
   (b) Prove that $\{w_1,\dots,w_{i-1},w_i-\beta w_j,w_{i+1},\dots,w_m\}$ for any $1\le i\neq j\le m$ and $\beta\in\mathbb{R}$ is again a basis of W. Find the matrix of L with respect to this new basis in terms of A.
   (c) Let A' be a matrix obtained from A by row reduction. Prove that A' is the matrix of L with respect to the bases $\{v_1,\dots,v_n\}$ of V and $\{w'_1,\dots,w'_m\}$ of W. Explain how $\{w'_1,\dots,w'_m\}$ is obtained from $\{w_1,\dots,w_m\}$.
   (d) We construct a basis $\{w'_1,\dots,w'_m\}$ as follows. Let $i_1>0$ be the smallest index such that $L(v_{i_1})\neq 0$ and define $w'_1=L(v_{i_1})$. Let $i_2>i_1$ be the smallest index such that $L(v_{i_2})\notin Span\{w'_1\}$ and define $w'_2=L(v_{i_2})$. Inductively, let $i_k>i_{k-1}$ be the smallest index such that $L(v_{i_k})\notin Span\{w'_1,\dots,w'_{k-1}\}$ and define $w'_k=L(v_{i_k})$. This way we get linearly independent $\{w'_1,\dots,w'_p\}$, which can be extended to a basis $\{w'_1,\dots,w'_p,\dots,w'_m\}$ of W. Prove that the matrix of L with respect to $\{v_1,\dots,v_n\}$ and $\{w'_1,\dots,w'_m\}$ is in echelon form.
   (e) Prove that if for any basis $\{w'_1,\dots,w'_m\}$ of W, the matrix of L with respect to $\{v_1,\dots,v_n\}$ and $\{w'_1,\dots,w'_m\}$ is in echelon form, then $\{w'_1,\dots,w'_m\}$ is constructed as in the previous problem. In particular, $\{w'_1,\dots,w'_p\}$ are uniquely determined by L and $\{v_1,\dots,v_n\}$.
   (f) Prove that the row reduction of A to a matrix A' in echelon form is unique.
(2) Consider a sequence of linear maps between vector spaces:

$$0\xrightarrow{d_0}V_1\xrightarrow{d_1}V_2\xrightarrow{d_2}\dots\xrightarrow{d_{k-1}}V_k\xrightarrow{d_k}0$$

This sequence is called *exact* if $Ker(d_i)=Img(d_{i-1})$ for all $1\le i\le k$. (Note that the zeroes at both ends are the zero vector spaces, and the maps d_0 and d_k are the obvious ones. Then the condition $Ker(d_1) = Img(d_0) = 0$ means that d_1 is injective, and $Img(d_{k-1}) = Ker(d_k) = V_k$ means that d_{k-1} is surjective.)

(a) Consider the short exact sequence ($k = 3$):

$$0 \to V_1 \overset{d_1}{\to} V_2 \overset{d_2}{\to} V_3 \to 0$$

Prove that $\dim V_2 = \dim V_1 + \dim V_3$. (Hint: identify the image and the kernel of d_2.)

(b) More generally, prove that the Euler characteristic of an exact sequence is zero:

$$\sum_i (-1)^i \dim V_i = 0$$

(Hint: prove that the following two sequences are exact:

$$0 \overset{d_0}{\to} V_1 \overset{d_1}{\to} V_2 \overset{d_2}{\to} \ldots \overset{d_{k-2}}{\to} Img(d_{k-2}) \to 0$$
$$0 \to Img(d_{k-2}) \to V_{k-1} \to V_k \to 0$$

and use induction on k.)

3. Optional – extra credit

(a) Problem 2.5.16 in the textbook.

(b) Problem 2.5.17 in the textbook.