

MATH 25A – PROBLEM SET #7
DUE FRIDAY NOVEMBER 22

1. PART A

- (1) Prove that the following are equivalent for an $n \times n$ matrix A :
- A has an inverse.
 - The columns of A are linearly independent.
 - The rows of A are linearly independent.
- (2) Recall that an inner product $\langle \cdot, \cdot \rangle$ on a vector space is a bilinear, symmetric function $V \times V \rightarrow \mathbb{R}$ satisfying the positivity condition: $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.

(a) Prove that

$$\langle f(x), g(x) \rangle = \int_{-1}^1 \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$$

defines an inner product on $\mathbb{R}[x]$, the space of polynomials in x .

(b) Let $T_n(x) = \cos(n \arccos(x))$ for $n = 0, 1, 2, \dots$. Prove that

$$T_{n+1} = 2xT_n(x) - T_{n-1}(x).$$

Deduce that T_n is a polynomial for all n . The polynomials T_n are called Chebyshev polynomials. (Hint: to prove the formula, find a relation between $\cos(a+b)$, $\cos(a-b)$, $\cos(a)$ and $\cos(b)$.)

(c) Prove that T_n for $n = 0, 1, 2, \dots$ form an orthogonal basis of $\mathbb{R}[x]$ with respect to the inner product given above.

(3) Recall that the dual V^* of a vector space V is

$$V^* = \text{Lin}(V, \mathbb{R}).$$

Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space V .

(a) Prove that we have a linear map $L : V \rightarrow V^*$, defined by

$$L(v)(w) = \langle v, w \rangle.$$

(Note: here $L(v)$ is an element of V^* , hence a function on V , and it can be applied to an element $w \in V$. You have to prove first that the map L is well-defined, that means $L(v)$ is a linear function $V \rightarrow \mathbb{R}$, and second that the map L is linear.)

(b) Prove that if V is finite dimensional, then L is an isomorphism. (Hint: what is the dimension of V^* ?)

(c) Let P_d be the set of polynomials in x of degree d or less. Prove that there exists a unique polynomial $p(x) \in P_d$ such that for any $q(x) \in P_d$ we have

$$\int_0^1 p(x)q(x)dx = q(3)$$

2. PART B

- (1) Problem 2.7.4 in the textbook.
(2) Problem 2.7.5(a) in the textbook.
(3) Problem 2.7.14(a),(b) in the textbook.