Math 25a Homework 1 Part A Solutions

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1a. If $x \in f(A \cap B)$, then there is an $s \in A \cap B$ such that $x = f(s)$. Since $s \in A$ and $s \in B$, we have $f(s) \in f(A)$ and $f(s) \in f(B)$. Thus $x \in f(A) \cap f(B)$, which proves $f(A \cap B) \subset f(A) \cap f(B)$.

1b. We make a sequence of equivalent statements: $x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in C \cap D \Leftrightarrow f(x) \in C$ and $f(x) \in D \Leftrightarrow x \in f^{-1}(C)$ and $x \in f^{-1}(D) \Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D)$. This proves $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

2. First, assume $A \subset B$. Then, we will prove $S - B \subset S - A$. If $x \in S - B$, then $x \notin B$, so $x \notin A$ (because A is a subset of B). Since $x \notin A$, $x \in S - A$. Hence all elements of $S - B$ are contained in $S - A$.

Conversely, assume $S - B \subset S - A$. Then, we will prove $A \subset B$. If $x \in A$, then $x \notin S - A$. Since $S - B \subset S - A$, we must have $x \notin S - B$. Then $x \in S - B$. That completes the proof.

Alternatively, to prove the converse, one can replace A by $S - A$ and B by $S - B$. Then $S - A \subset S - B$ implies $S - (S - A) \subset S - (S - B)$, and it is easy to verify that $S - (S - A) = A$ and $S - (S - B) = B$.

3. First, we prove (a) implies (b). Suppose $f^{-1}(t)$ contains x and y. Then $f(x) = f(y)$. Since f is injective, $x = y$. Therefore, $f^{-1}(t)$ contains at most one element.

Next, we prove (b) implies (c). We have $f(A \cap B) \subset f(A) \cap f(B)$ from problem (1a), so it suffices to show $f(A) \cap f(B) \subset f(A \cap B)$. Let x be an element of $f(A) \cap f(B)$. Thus, $x = f(a)$ and $x = f(b)$ for some $a \in A$, $b \in B$. Because $f^{-1}(x)$ must contain a and b, by (b) we have $a = b$. Then $a \in A \cap B$ and so $x = f(a) \in f(A \cap B)$. This proves $f(A) \cap f(B) \subset f(A \cap B)$.

Finally, we prove (c) implies (a). We prove the contrapositive. If f is not injective, then there exist $a, b \in S$ such that $a \neq b$ and $f(a) = f(b)$. Let $A = \{a\}$ and $B = \{b\}$. Then $f(A \cap B) = \emptyset$ but $f(A) \cap f(B) = \{f(a)\}$, which contradicts (c). Therefore, our proof is complete.

4a. Using a base case of $n = 1$, we easily verify that the relation holds.

Assuming the equation is true for n, we find

$$\sum_{i=1}^{n+1} i^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$$

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

Therefore, our induction hypothesis holds for $n+1$, and the induction is complete.

4b. Using a base case of $n = 1$, we easily verify that the relation holds. Assuming the equation is true for n, we find

$$\sum_{i=1}^{n+1} 3^{2i-1} = \frac{3(9^n - 1)}{8} + 3^{2n+1}$$

$$= \frac{3 \cdot 9^n - 3 + 8 \cdot 3^{2n+1}}{8}$$

$$= \frac{3 \cdot 9^n - 3 + 24 \cdot 9^n}{8}$$

$$= \frac{27 \cdot 9^n - 3}{8}$$

$$= \frac{3(9^{n+1} - 1)}{8}$$

Therefore, our induction hypothesis holds for $n+1$, and the induction is complete.

5. a divides b (often written $a|b$) if and only if $b = ka$ for some $k \in \mathbb{Z}$.

Reflexivity: $s \sim s \Leftrightarrow 12$ divides 0, which is true.

Symmetry: $s \sim t \Rightarrow s - t = 12k \Rightarrow t - s = (-12)k$. Thus $t - s$ is divisible by 12, so $t \sim s$.

Transitivity: $s \sim t \Leftrightarrow s - t = 12k$ and $t \sim u \Leftrightarrow t - u = 12m$, with $k, m \in \mathbb{Z}$. Adding these equations gives $s - u = 12(k+m)$, so $s \sim t$ and $t \sim u$ implies $s \sim u$, as desired.

6. If $\overline{a} \cap \overline{b} \neq \emptyset$, we are done, so suppose $\overline{a} \cap \overline{b} \neq \emptyset$. Then there exists an x in both $\overline{a}$ and $\overline{b}$. By definition, we have $a \sim x$ and $b \sim x$. Then $a \sim b$. For all $y \in \overline{a}$, we have $y \sim a$, so $y \sim b$. That means $y \in \overline{b}$. Similarly, one can show for all $z \in \overline{b}$, $z \in \overline{a}$. That means $\overline{a} = \overline{b}$, and our proof is complete.

7a. Reflexivity: $ab = ab$ so $(a, b) \sim (a, b)$.

Symmetry: If $(a,b) \sim (c,d)$, then $ad = bc$. Then $bc = ad$, so $(c,d) \sim (a,b)$.

Transitivity: If $(a,b) \sim (c,d)$, then $ad = bc$. Dividing by bd, which is not 0, $\frac{a}{b} = \frac{c}{d}$. Similarly, $(c,d) \sim (e,f)$ implies $\frac{c}{d} = \frac{e}{f}$. Then $\frac{a}{b} = \frac{e}{f}$, so $af = be$ and $(a,b) \sim (e,f)$, which implies transitivity.

7b. We claim that this set is identical to the set of rational numbers $\mathbb{Q}$. Identify $\overline{(a,b)} \in S/\sim$ with $\frac{a}{b}$, noting that $b \neq 0$. As in part (7a), we find $(a,b) \sim (c,d) \Leftrightarrow \frac{a}{b} = \frac{c}{d}$, so $\overline{(a,b)}$ represents a unique rational number. Finally, since $\frac{x}{y}$ is represented by $\overline{(x,y)}$, every rational is identified with an element of $S/\sim$. Therefore, $S/\sim$ is identical to $\mathbb{Q}$.

8a. Reflexivity: $f(s) = f(t)$ so $s \sim t$.

Symmetry: $s \sim t \Rightarrow f(s) = f(t) \Rightarrow f(t) = f(s) \Rightarrow t \sim s$.

Transitivity: $s \sim t$ and $t \sim u$ implies $f(s) = f(t)$ and $f(t) = f(u)$, from which we conclude $f(s) = f(u)$ and $s \sim u$.

8b. For $\overline{a} \in S/\sim$, define $\overline{f}(\overline{a}) = f(a)$.

First, we show this map is well-defined. If $\overline{a} = \overline{b}$, then $a \sim b$. Then $f(a) = f(b)$, so $\overline{f}(\overline{a}) = \overline{f}(\overline{b})$, as desired.

Next, we show the map is injective. If $\overline{f}(\overline{a}) = \overline{f}(\overline{b})$, then $f(a) = f(b)$. Hence, $a \sim b$, and $\overline{a} \cap \overline{b} \neq \emptyset$ because a and b are in both sets. By problem (6), $\overline{a} = \overline{b}$. Therefore, $\overline{f}$ is injective.