PROBLEM SET #1 SOLUTIONS
PART B
September 27, 2002

(1) (a) Let $\varepsilon > 0$. Since $\{a_i\}$ converges to a, for every $\varepsilon/2$ there exists a natural number N_1 such that $|a_i - a| < \varepsilon/2$ for all $i \geq N_1$. Similarly there exists an $N_2 \in \mathbb{N}$ such that $|b_i - b| < \varepsilon/2$ for all $i \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then we have for all $i \geq N$

$$|(a_i + b_i) - (a + b)| \leq |a_1 - a| + |b_1 - b| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so that $\{a_i + b_i\} \to a + b$.

(b) Let $\varepsilon > 0$. Then since $\{a_i\}$ converges to a, for every $\varepsilon/2 > 0$ there exists a $N_1 \in \mathbb{N}$ such that $|a_i - a| < \varepsilon/2$ for all $i \geq N_1$. Similarly for every $\varepsilon/2 > 0$ there exists a $N_2 \in \mathbb{N}$ such that $|a_i - b_i(-0)| = |b_1 - a_1| < \varepsilon/2$ for every $i \geq N_2$. Let $N = max\{N_1, N_2\}$. Then for all $i \geq N$ we have

$$|b_i - a| \leq |b_i - a_i| + |a_i - a| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and so we see that $\{b_i\} \to a$ as well.

Remark. Although no one wrote this, it is also possible to prove (b) directly from (a), if one considers the sum of the sequence $\{-a_i + b_i\} \to 0$ and the sequence $\{a_i\} \to a$.

(2) (a) Since $\{a_i\} \to a$ we have that for any $\varepsilon/2 > 0$ there is a $N_1 \in \mathbb{N}$ so that $|a_i - a| < \varepsilon/2$ for all $i \geq N_1$. Now consider the quantity

$$|b_i - a| = \left| \frac{a_1 + \cdots + a_i}{i} - a \right|$$

$$\leq \left| \frac{(a_1 - a) + \cdots + (a_{N_1-1} - a)}{i} \right| + \left| \frac{(a_{N_1} - a) + \cdots + (a_i - a)}{i} \right|$$

Claim. $\frac{(a_1-a)+\cdots+(a_{N_1-1}-a)}{i} \to 0$ *as* $i \to \infty$, *i.e. for every* $\varepsilon/2K > 0$ *there exists an* $N_2 \in \mathbb{N}$ *so that*

$$\left| \frac{(a_1 - a) + \cdots + (a_{N_1-1} - a)}{i} \right| < \frac{\varepsilon}{2K}$$

for all $i > N_2$.

Proof. In particular, let $K = (a_1 - a) + \cdots + (a_{N_1-1} - a)$. Note that K is fixed. Then we see that, if K/i is to be less than $\varepsilon/2K$, we may simply take N_2 to be the smallest integer greater than $2/\varepsilon$. Such an N_2 exists for all $\varepsilon > 0$, proving the claim. □

Now let $N = \max\{N_1, N_2\}$. Then we have, for $i > N$,

$$|b_i - a| \leq \left| \frac{(a_1 - a) + \cdots + (a_{N_1-1} - a)}{i} \right| + \left| \frac{(a_{N_1} - a) + \cdots + (a_i - a)}{i} \right|$$

$$< K \cdot \frac{\varepsilon}{2K} + \frac{i - N_1}{i} \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so that $\{b_i\} \to a$, as desired.

(b) Consider the sequence $a_i = (-1)^{i+1}$. This sequence does not converge, as for any choice of potential limit point a, $|a_i - a| \geq 1$ for infinitely many i. However, if we note that $|\sum_{i=1}^{n} a_i| \leq 1$ for all n we see that, for $\varepsilon > 0$ and $n > 1/\varepsilon$,

$$|b_i - 0| \leq |1/n| \leq \varepsilon$$

so that the sequence $\{b_i\}$ converges to zero.

(3) Let $X \subset \mathbb{R}$ be a nonempty set that is bounded from above. For completeness we transcribe the strategy outlined on p. 19 of Hubbard and Hubbard; this is, until the end of the proof, exactly the same as the proof we wish to complete here.

"Let us suppose that $x \in X$ is an element (which we know exists, since $X \neq \emptyset$) and that a is an upper bound. We will assume that $x > 0$ (the case $x \leq 0$ is slightly different). If $x = a$, we are done: the least upper bound is a.

"If $x \neq a$, there is a first digit j such that the jth digit of x is smaller than the jth digit of a. Consider all the numbers in $[x, a]$ that can be written using only j digits after the decimal, than all zeroes. This is a finite non-empty set. In fact, it has at most 10 elements and $[a]_j$ is one of them[1]. Let b_i be the largest which is not an upper bound. Now consider the set of numbers in $[b_j, a]$ that have only $j + 1$ digits after the decimal point, than all zeroes. Again this is a finite non-empty set, so you you can choose the largest which is not an upper bound; call it b_{j+1}. It should be clear that b_{j+1} is obtained by adding one digit to b_j. Keep going this way, defining numbers $b_{j+2}, b_{j+3}, \cdots$, each time adding one digit to the previous number."

Note that the sequence $\{b_j\}$ constructed in this manner is monotone increasing and bounded above (by a). Therefore by property C2 this sequence converges to some limit point b; this point b has the property that its kth digit equals b_k. We will show that b is the least upper bound for X.

Remark. It is possible to use similar means to construct other monotone increasing bounded from above (and hence convergent) sequences in X, but these need not converge to $\sup(X)$. It is only because we were very careful in designing this particular sequence that it converges to the value of interest.

First note that b is an upper bound, since if there $y \in X$ with $y > b$, then there exists a first digit k greater than the kth digit of b, in which case b_k was not the largest element of X with k digits which is not the upper bound, contrary to our definition of b_k. Now suppose that there exists another upper bound b' with $b' < b$. Then there is a first digit k' such that the k'th digit of b' differs from the k'th digit of b. But then $b_{k'}$ would be greater than b', and since $b_{k'}$ is by supposition not an upper bound, b' cannot be an upper bound either. Therefore b is the least upper bound for X.

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[1]$[a]_j$ denotes the finite decimal consisting of all the digits of a before the decimal, and j digits after the decimal

(4) Let $\{a_i\}$ be a sequence converging to a. Then for every $\varepsilon/2 > 0$ there exists an $N \in \mathbb{N}$ such that $|a_i - a| < \varepsilon/2$ for all $i > N$. It follows that, for $i, j > N$,

$$
\begin{aligned}
|a_i - a_j| &= |a_i - a + a - a_j| \\
&\leq |a_i - a| + |a_j - a| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

so that $\{a_i\}$ is in fact a Cauchy sequence.

(5) Fix $\varepsilon = 1$ and let N be the natural number such that $|a_i - a_j| < 1$ for all $i, j > N$. Fix $j = N+1$. Then $|a_i| = |a_i - a_{N+1} + a_{N+1}| \leq |a_i - a_{N+1}| + |a_{N+1}| < 1 + |a_{N+1}|$ for all $i > N$, by definition of a Cauchy sequence. Let $M = \max\{|a_1|, \ldots, |a_N|, |a_{N+1}| + 1\}$. Since this is a finite set such an M is well-defined, and by definition $|a_i| \leq M$ for all i, so that $\{a_i\}$ is bounded.

(6)   (a) From the previous exercise we know that $\{a_i\}$ is bounded below. Furthermore, the sequence $\{b_i\}$ is monotone decreasing: if $b_i = \sup\{a_i, a_{i+1}, \ldots\}$, then $b_{i+1} < b_i$, and otherwise the suprema of the two sequences are the same, so that $b_{i+1} = b_i$. Thus $\{b_i\}$ is a monotone decreasing sequence in $\mathbb{R}$ that is bounded below, and so by property C2 (applied to $\{-b_i\}$) it has a limit b. Note that it is possible to do this problem without invoking the $\varepsilon - N$ definition of a limit.

   (b) We will consider the sequence $\{b_i - a_i\} = \sup\{0, a_{i+1} - a_i, a_{i+2} - a_i, \ldots\}$. Since $\{a_i\}$ is a Cauchy sequence, we know that for every $\varepsilon/2 > 0$ there is an $N \in \mathbb{N}$ such that $|a_{i+j} - a_i| < \varepsilon/2$ for all $i > N$ and $j > 0$ (here we are simply rewriting the definition using slightly different notation). Therefore for $i > N$ we see that

$$
|b_i - a_i| = |\sup\{0, a_{i+1} - a_i, a_{i+2} - a_i, \ldots\}| \leq \varepsilon/2 < \varepsilon
$$

   so that $\{b_i - a_i\}$ converges to zero $\Rightarrow \{a_i\}$ converges to b, by Exercise (1)(b).

(7) We will show that C2 $\Rightarrow$ C3 and that C3 $\Rightarrow$ C1 ($\Leftrightarrow$ C2).

   C2 $\Rightarrow$ C3. Suppose that C2 holds and let $\{a_i\}$ be an arbitrary Cauchy sequence in $\mathbb{R}$. Then by (5) $\{a_i\}$ is bounded from above and as in (6) we can construct a monotone increasing sequence $\{b_i\}$, also bounded from above. By assumption this sequence converges to some number b; by (6)(b), then, we see that $\{a_i\}$ converges as well.

   C3 $\Rightarrow$ C1. Let X be a bounded subset of $\mathbb{R}$, and consider the sequence (b_i) constructed in exercise (3). We will show that $\{b_i\}$ is Cauchy. Fix $\varepsilon > 0$ and choose N so that $10^{-N} < \varepsilon$. Then if we examine the definition of the b_i we see that $|b_i - b_j| \leq 10^{-N} < \varepsilon$ for all $i, j > N$, as desired. By hypothesis C3, then, (b_i) converges to some limit point b, which as in problem (3) is the least upper bound of X.

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