Mathematics 25a Fall 2002

# Problem Set #2 Solutions
## Part A
*October 4, 2002*

*Note.* Following the convention in higher mathematics, I will denote vectors by $x, y, \ldots$ rather than by $\vec{x}, \vec{y}, \ldots$.

(1) (a) To show that $W \subset V$ is a subspace we need only verify that it is closed under addition and scalar multiplication (this is a generalization of Definition 1.1.5 on p. 39 of Hubbard and Hubbard).

*Closure under addition:* Let $x, y \in W_1 \cap W_2$. Since $x$ and $y$ are in $W_1$ and $W_1$ is a subspace of $V$, $x + y \in W_1$. Similarly, since $x$ and $y$ are in $W_2$ and $W_2$ is a subspace of $V$, $x + y \in W_2$. Therefore $x + y \in W_1 \cap W_2$.

*Closure under scalar multiplication:* Let $\lambda \in \mathbb{R}$ and let $x \in W_1 \cap W_2$. Since $x \in W_1$ and $W_1$ is a subspace, $\lambda x \in W_1$, and since $x \in W_2$ and $W_2$ is a subspace, $\lambda x \in W_2$, so we see that $\lambda x \in W_1 \cap W_2$ as well.

(b) ($\Rightarrow$) Suppose that $W_1 \cup W_2$ is a subspace of $V$, and suppose that neither $W_1$ nor $W_2$ is contained in the other. Then there exists $x \in W_1$, $x \notin W_2$ and $y \in W_2$, $y \notin W_1$. Consider the vector $x + y$. Now, since $W_1 \cup W_2$ is a subspace, $x + y$, being the sum of two elements of $W_1 \cup W_2$, must be an element of $W_1 \cup W_2$. However, $x + y$ cannot be in $W_1$: since $W_1$ is closed under addition and scalar multiplication then $(x + y) + (-1)x = y \in W_1$, a contradiction. Similarly $x + y$ cannot be in $W_2$, for that would imply that $(x + y) + (-1)y = x \in W_2$, again a contradiciton. Therefore either $W_1 \subset W_2$ or $W_2 \subset W_1$.

($\Leftarrow$) Suppose that $W_1 \subset W_2$ (the proof is exactly the same for $W_2 \subset W_1$). Then $W_1 \cup W_2 = W_2$, and so is a subspace of $V$ by our initial assumption.

(2) As in the previous problem, we show that each of these subsets is closed under addition and scalar multiplication.

*Remark.* The image of $T$, $T(V)$, is often denoted by $\mathrm{Im}(T)$, which I use here.

(a) *Closure under addition:* Let $w_1, w_2 \in \mathrm{Im}(T)$. Then there exist $w_1, w_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Since $V$ is a vector space $(v_1 + v_2) \in V$, and by linearity of $T$, $T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2 \in \mathrm{Im}(T)$.

*Closure under scalar multiplication:* Let $w = T(v) \in \mathrm{Im}(V)$ and let $\lambda \in \mathbb{R}$. Then since $\lambda v \in V$ and $T$ is linear we have $T(\lambda v) = \lambda T(v) = \lambda w \in \mathrm{Im}(T)$, and so $\mathrm{Im}(T)$ is a subspace of $W$.

(b) *Closure under addition:* Let $x, y \in Ker(T)$, so that $T(x) = T(y) = 0$. Then by linearity of $T$, $T(x + y) = T(x) + T(y) = 0 + 0 = 0 \Rightarrow (x + y) \in Ker(T)$.

*Closure under scalar multiplication:* Let $x \in Ker(T)$ and $\lambda \in \mathbb{R}$. Then by linearity of $T$ $T(\lambda x) = \lambda T(x) = \lambda \cdot 0 = 0 \Rightarrow \lambda x \in Ker(T)$, proving that $Ker(T)$ is a subspace of $V$.

(3) (a) Define addition and scalar multiplication in the "natural" way: for $v \in V, \lambda \in \mathbb{R}$ and $S, T \in Lin(V, W)$, let

$$(S+T)(v) = S(v) + T(v) \qquad \text{and} \qquad (\lambda T)(v) = \lambda \cdot T(v).$$

We now verify the vector space axioms for $Lin(V, W)$. In what follows let $R, S, T \in Lin(V, W)$, $\lambda, \mu \in \mathbb{R}$, and $v \in V$.

(i) (Existence of zero) Let $0 \in Lin(V, W)$ be the map sending all elements of $V$ to 0. It is clear that 0 is linear, and for any $v$ and $T$, $(0+T)(v) = 0(v) + T(v) = 0 + T(v) = T(v)$, so that $0 + T = T$.

(ii) (Existence of additive inverse) For all $T$ let $(-T)$ be defined by $(-T)(v) = -(T(v))$, the additive inverse of $T(v)$ in $W$, for all $v \in V$. It is clear from the definition that $-T \in Lin(V, W)$. To see that $(-T)$ does in fact behave as an additive inverse consider that $(T + (-T))(v) = T(v) + (-T)(v) = T(v) + -(T(v)) = 0 \; \forall \; v \Rightarrow (T + (-T)) = 0$.

(iii) (Commutativity under addition) $(S+T)(v) = S(v) + T(v) = T(v) + S(v) = (T+S)(v)$, by commutativity under addition in $W$.

(iv) (Associativity under addition) Since addition is associative in $W$. we have

$$\begin{aligned} ((R+S)+T)(v) &= (R+S)(v) + T(v) \\ &= R(v) + S(v) + T(v) \\ &= R(v) + (S(v) + T(v)) \\ &= R(v) + (S+T)(v) \\ &= (R + (S+T))(v) \end{aligned}$$

so that $R + (S+T) = (R+S) + T$, as desired.

(v) (Existence of multiplicative identity) In this case the multiplicative identity is simply the multiplicative identity 1 in $W$ (i.e. the real number 1), for then $(1T)(v) = 1 \cdot T(v) = T(v) \; \forall \; T$.

(vi) (Associativity under multiplication) By associativity of multilpication in $W$, we have $(\lambda(\mu T))(v) = \lambda \cdot (\mu T)(v) = \lambda\mu \cdot T(v) = (\lambda\mu) \cdot T(v) = ((\lambda\mu)T)(v)$.

(vii) (Distributivity under scalar addition) By distributivity under scalar addition in $W$, we have $((\lambda+\mu)T)(v) = (\lambda+\mu)(T(v)) = \lambda \cdot T(v) + \mu \cdot T(v) = (\lambda T + \mu T)(v)$.

(viii) (Distributivity under vector addition) By distributivity under vector addition in $W$, we have $(\lambda(S+T))(v) = \lambda \cdot (S+T)(v) = \lambda \cdot S(v) + \lambda \cdot T(v) = (\lambda S + \lambda T)(v)$.

(b) As shown in class, the linear transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$ can all be represented by $m \times n$ matrices. Thus $(\mathbb{R}^n)^* = Lin(\mathbb{R}^n, \mathbb{R})$ is the set of all $1 \times n$ matrices with real entries. In other words, if $\mathbb{R}^n$ is thought of as the set of all $n$ dimensional column vectors with real entries, $(\mathbb{R}^n)^*$ is the set of all $n$ dimensional row vectors with real entries (hence–in part–the term *dual* to describe it).

(4) Let $v_1, v_2 \in V$ and $w_1, w_2 \in W$, and let $\lambda \in \mathbb{R}$. Denote elements of $V \times W$ as ordered pairs $(v, w)$, where the first entry always denotes an element of $V$ and the second always denotes an element of $W$. Then we define addition and scalar multiplication by letting

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \qquad \text{and} \qquad \lambda(v, w) = (\lambda v, \lambda w).$$

Note that these operations are well-defined.

We now verify that these operations make $V \times W$ into a vector space. In what follows let $v, v_1, v_2, v_3 \in V, w, w_1, w_2, w_3 \in W$, and $\lambda, \mu \in \mathbb{R}$. We will also assume–and make liberal use of–the vector space properties of $V$ and $W$.

(i) (Existence of zero) Consider the element $(0, 0)$. Then $(0, 0) + (v, w) = (0 + v, 0 + w) = (v, w)$, so that $(0, 0)$ indeed functions as an additive identity.

(ii) (Existence of additive inverse) Let $-(v, w) = (-v, -w)$. Then $(v, w) + (-(v, w)) = (v + (-v), w + (-w)) = (0, 0)$, so $(-v, -w)$ is the required additive inverse of $(v, w)$.

(iii) (Commutativity under addition) $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$.

(iv) (Associativity under addition) Here we have

$$\begin{aligned}((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) \\ &= (v_1, w_1) + ((v_2, w_2) + (v_3, w_3)).\end{aligned}$$

(v) (Existence of multiplicative identity) The multiplicative identity is again the real number 1, which we verify by observing that $1(v, w) = (1v, 1w) = (v, w)$.

(vi) (Associativity under multiplication) $\lambda(\mu(v, w)) = \lambda(\mu v, \mu w) = (\lambda\mu v, \lambda\mu w) = ((\lambda\mu)v, (\lambda\mu)w) = (\lambda\mu)(v, w)$.

(vii) (Distributivity under scalar addition) $(\lambda + \mu)(v, w) = ((\lambda + \mu)v, (\lambda + \mu)w) = (\lambda v + \mu v, \lambda w + \mu w) = (\lambda v, \lambda w) + (\mu v, \mu w) = \lambda(v, w) + \mu(v, w)$.

(viii) (Distributivity under vector addition) $\lambda((v_1, w_1) + (v_2, w_2)) = \lambda(v_1 + v_2, w_1 + w_2) = (\lambda(v_1 + v_2), \lambda(w_1 + w_2)) = (\lambda v_1 + \lambda v_2, \lambda w_1 + \lambda w_2) = (\lambda v_1, \lambda w_1) + (\lambda v_2, \lambda w_2) = \lambda(v_1, w_1) + \lambda(v_1, w_2)$.

(5) (a) We need to show that $\sim$ is reflexive, transitive, and associative. In what follows let $u, v, w \in V$.

*Reflexivity:* Since $W$ is a subspace, $0 \in W$. Therefore $v - v = 0 \in W$, i.e. $v \sim v$.

*Transitivity:* If $v \sim w$, then $v - w = 0 \Rightarrow w - v = 0$, so that $w \sim v$.

*Associativity:* If $u \sim v$ and $v \sim w$, then $u - v = v - w = 0 \Rightarrow (u - v) + (v - w) = u - w = 0$, and so $u \sim w$.

(b) We first show that addition and multiplication are well-defined. Let $[v_1]$ and $[v_2]$ be equivalence classes represented by $v_1$ and $v_2$, respectively. Let $v_1', v_2'$ be two other elements in $[v_1]$ and $[v_2]$, and let $\lambda \in \mathbb{R}$. We need to show that $(v_1 + v_2) \sim (v_1' + v_2')$, so that $[v_1] + [v_2] = [v_1 + v_2]$, and that $\lambda v_1 \sim \lambda v_1'$, so that $\lambda[v_1] = [\lambda v_1]$. Since $v_1' - v_1 \in W$

and $v_2' - v_2 \in W$, we see that $(v_1' + v_2') - (v_1 + v_2) = (v_1' - v_1) + (v_2' - v_2) \in W$, so $(v_1' + v_2') \sim (v_1 + v_2)$, i.e $(v_1' + v_2') \in [v_1 + v_2]$.
To check multiplication, note that if $v_1 \sim v_1'$ then $\lambda v_1 - \lambda v_1' = \lambda(v_1 - v_1') \in W$, so that $\lambda v_1 \sim \lambda v_1'$ and $\lambda v_1' \in [\lambda v_1]$. Thus addition and multiplication on equivalence classes is well-defined by considering addition and multiplication on representative elements of each equivalence class. With that we see that the axioms of a vector space are trivially satisfied by considering the vector space structure on the representative elements $v_i$ of the elements of $V/W$.

(6) (a) This is an exercise in notation. Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times p$ matrix. Let the $(i, j)$th element $(A)_{ij} = a_{ij}$ and $(B)_{ij} = b_{ij}$; then $(A^t)_{ij} = a_{ji}$ and $(B^t)_{ij} = b_{ji}$, and we see that

$$\begin{aligned} ((A \cdot B)_{ij})^t &= \left( \sum_{k=1}^{n} a_{ik} b_{kj} \right)^t \\ &= \sum_{k=1}^{n} a_{jk} b_{ki} \\ &= (B^t \cdot A^t)_{ij}. \end{aligned}$$

Since this equality holds for every $i$ and $j$ running though the elements of $A$ and $B$, we see that $(A \cdot B)^t = B^t \cdot A^t$.

(b) Suppose that $A$ is invertible, i.e that there exists a matrix $A^{-1}$ with the property that $A \cdot A^{-1} = A^{-1} \cdot A = I$.

**Claim.** $(A^t)^{-1} = (A^{-1})^t$.

*Proof.* First note that because $A^{-1}$ exists for all invertible $A$, the matrix $(A^{-1})^t$ is always well-defined. We will simply check that $(A^{-1})^t$ is indeed the two-sided multiplicative inverse for $A^t$. Using the multiplicative property proved in the previous part of this exercise and the fact that $I^t = I$, we see that

$$A^t \cdot (A^t)^{-1} = A^t \cdot (A^{-1})^t = (A^{-1} \cdot A)^t = I^t = I$$

and

$$(A^t)^{-1} \cdot A^t = (A^{-1})^t \cdot A^t = (A \cdot A^{-1})^t = I^t = I$$

as desired. □

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