Math 25a Homework 3 Part A Solutions

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For each of the parts of problem 1, let S be the set in question.

1a. This is open and not closed.

Note that S may be defined as $\{x \in \mathbb{R}^2 \mid 1 < d(x,0) < \sqrt{2}\}$. Choose a positive $r < \min\{d(x,0)-1, \sqrt{2}-d(x,0)\}$. The open ball $B_r(x)$ lies entirely within S because for any $b \in B_r(x)$, $d(b,0) \leq d(b,x)+d(x,0) \leq r+d(x,0) < \sqrt{2}$ and $d(b,0) \geq d(x,0) - d(b,x) \geq d(x,0) - r > 1$. So, S is open.

S is not closed because any ball around $(1,0)$ will contain points $(1+\epsilon, 0)$ for small positive ϵ, which lie within S.

1b. This is open and not closed.

Given a point $(x,y) \in S$, choose a positve $r < \min\{|x|,|y|\}$. Then $B_r(x,y)$ will lie in S because for any $b \in B_r(x,y)$, its x- and y-coordinates will be positive by how we chose r. This S is open.

S is not closed because any ball around 0 will contain points of the form (ϵ, ϵ) for small positive ϵ, which are not in S.

1c. This set is closed and not open.

Its complement is open because given any point (x,y) in the complement of S, for $r < |y|$, the ball $B_r(x,y)$ will lie entirely in S. This is because the y-coordinate of any point in $B_r(x,y)$ is positive by how we chose r.

S is not open because any ball around 0 contains points of the form $(\epsilon, 0)$ for small positive ϵ, which do not lie in S.

1d. This set is neither closed nor open.

$\mathbb{Q}$ is not open because any ball around 0 contains, for example, a point of the form $1/n$ for n large enough. Its complement is also not open because any ball around 0 contains, for example, the irrational number $\sqrt{2}/n$ for n large enough.

2a. This means that as (x,y) approaches $(0,0)$, $f(x,y)$ approaches the value a, even though f is not defined at a. Specifically, for any $\epsilon > 0$, there exists a δ so that $|(x,y)| < \delta$ implies $|f(x,y) - a| < \epsilon$.

2b. (Rather than appealing directly to the $\epsilon - \delta$ definition of a limit, we are going to use basic properties of limits, so this proof won't get too messy.) $\lim_{(x,y)\to 0} \frac{\sin(x+y)}{\sqrt{x^2+y^2}}$ does not exist. Taking $y = 0$, $x \to 0$, the limit becomes $\lim_{x\to 0} \frac{\sin x}{|x|}$. Using L'Hospital's rule or otherwise, one can find that this limit is 1 from the right and -1 from the left. Since a limit cannot converge to two values, it does not exist.

$\lim_{(x,y)\to 0}(|x| + |y|)\ln(x^2 + y^4)$ exists and is equal to 0. It is possible to prove this using a substitution (e.g. $(x, y) = (r\cos\theta, r\sin\theta)$) and L'Hospital's rule. However, this is really messy, so I'll present a cleaner, but less apparent, way of calculating the limit.

We will use the inequality $x^2 + y^2 \geq \frac{1}{2}(|x| + |y|)^2$. This is true because it is equivalent to $(|x| - |y|)^2 \geq 0$.

We assume $|x|, |y| < 1/2$. Then, $x^2+y^4 > x^4+y^4 \geq \frac{1}{2}(x^2+y^2)^2 \geq \frac{1}{2}(\frac{1}{2}(|x|+|y|)^2)^2 = \frac{1}{8}(|x| + |y|)^4$. Since the natural log is an increasing function, this implies $\ln(x^2 + y^4) > \ln(\frac{1}{8}(|x| + |y|)^4)$. These are negative, so $|\ln(x^2 + y^4)| < |\ln(\frac{1}{8}(|x| + |y|)^4)|$. Multiply by $|x| + |y|$ to get $|(|x| + |y|)\ln(x^2 + y^4)| < |(|x| + |y|)\ln(\frac{1}{8}(|x| + |y|)^4)|$. Let $t = |x| + |y|$. We know $t \to 0$ as $(x, y) \to 0$. Substituting, $|(|x| + |y|)\ln(\frac{1}{8}(|x| + |y|)^4)| = |t\ln\frac{1}{8}t^4| = |4t\ln\frac{t}{8}|$. Good old calculus tells us that as $t \to 0$, this approaches 0.

By our inequality $|(|x| + |y|)\ln(x^2 + y^4)| < |(|x| + |y|)\ln((|x| + |y|)^4)|$, as $(x, y) \to 0$, we must have $|(|x| + |y|)\ln(x^2 + y^4)| \to 0$. The only possibility is that the limit is 0.

3. See theorem 1.5.27 of your text (page 103).

4. This is theorm 1.5.29 or your text (page 103). It is the direct result of theorem 1.5.24 and 1.5.27 (page 101-103).

For another proof, we have for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \delta$ implies $|g(f(x)) - g(f(a))| < \epsilon$. Since f is continuous, there exists δ' such that $|x - a| < \delta'$ implies $|f(x) - f(a)| < \delta$. Putting it together, given an $\epsilon > 0$, we have a δ' so that $|x-a| < \delta'$ implies $|g(f(x))-g(f(a))| < \epsilon$, which proves the continuity of $g \circ f$.

5. f is not continuous at any rational point a. Given any $\epsilon > 0$ and any $\delta > 0$, there is an irrational x such that $|x-a| < \delta$. Then, $|f(a)-f(x)| = f(a)$ is not less than ϵ for $\epsilon > f(a)$.

f is continuous at any irrational point a. Given ϵ, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Observe that there are only finitely many rational numbers in the interval $(a - 1, a + 1)$ with denominator at most N. Let δ be the distance between a and the closest of these numbers. Thus, all rationals with denominator at most N lie outside $(a - \delta, a + \delta)$. Then, for all x such that $|x - a| < \delta$ (i.e. x in $(a - \delta, a + \delta)$), $|f(a) - f(x)| = f(a)$ is at most $\frac{1}{N} < \epsilon$. That is the definition of continuity at a.

6. We will show f is continuous at the point $c \in (a, b)$. Select $x \in [a, c]$. Set $\lambda = \frac{x-a}{c-a}$; then it is easy to check $0 \leq \lambda \leq 1$. Thus, applying the definition of convexity with $u = c$ and $v = a$, we get $f(\lambda c + (1-\lambda)a) \leq \lambda f(c) + (1-\lambda) f(a)$. Simplifying, one finds $f(x) - f(a) \leq \frac{f(c)-f(a)}{c-a}(x-a)$ for $x \in [a, c]$. Similarly, one finds $f(x) - f(c) \leq \frac{f(c)-f(b)}{c-b}(x-c)$ for $x \in [c, b]$.

Plug in $u = a$, $v = x$, and $\lambda = \frac{x-c}{x-a}$ for $x \in [c, b]$ to the equation for convexity and some algebra will give you $f(x) - f(a) \geq \frac{f(c)-f(a)}{c-a}(x-a)$. Similarly, one finds $f(x) - f(c) \geq \frac{f(c)-f(b)}{c-b}(x-c)$ for $x \in [a, c]$.

Putting the results together, we see that $f(x)$ lies between the two lines $f(x) - f(c) = \frac{f(c)-f(b)}{c-b}(x-c)$ and $f(x) - f(a) = \frac{f(c)-f(a)}{c-a}(x-a)$. The latter equation is equivalent to $f(x) - f(c) = \frac{f(c)-f(a)}{c-a}(x-c)$. Thus, $|f(x) - f(c)| \leq \max\left\{\left|\frac{f(c)-f(b)}{c-b}(x-c)\right|, \left|\frac{f(c)-f(a)}{c-a}(x-c)\right|\right\}$. Given any $\epsilon > 0$, choose $\delta < \min\left\{\left|\epsilon\frac{c-b}{f(c)-f(b)}\right|, \left|\epsilon\frac{c-a}{f(c)-f(a)}\right|\right\}$. Then, $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.