Mathematics 25a Fall 2002

# Problem Set #3 Solutions
## Part B
*October 11, 2002*

(1) Let $d(v,w) = \|v-w\|$. We will show that $d$ satisifes the properties of a metric on $V$. In what follows let $u,v,w$ be arbitrary elements of $V$, except for where specified.
- $\|u\| \geq 0 \ \forall\ u \in V$, and in particular for $u = v-w$, so $d(v,w) \geq 0$ for any $v,w \in V$. Furthermore $\|u\| = 0$ if and only if $u = 0$, so that in particular $\|v-w\| = d(v,w) = 0$ if and only if $v = w$.
- $\|\alpha u\| = |\alpha|\|u\|$; letting $\alpha = -1$ gives $\|-u\| = |-1|\|u\| = \|u\|$, so that $\|v-w\| = \|w-v\| \Rightarrow d(v,w) = d(w,v)$.
- Let $p,q,r \in V$ and let $v = p-q$, $w = q-r$. Then $\|v+w\| = \|p-r\| \leq \|p-q\| + \|q-r\|$, so that $d(p,r) \leq d(p,q) + d(q,r)$.

(2) We will consider each metric in turn. In each case we take $d(p.q) = \|p-q\|$, or equivalently $d(v,0) = \|v\|$. The fact that $\|v\| \geq 0$ for all $v \in V$ and that $\|v\| = 0 \Leftrightarrow v = 0$ follow from the properties of the induced metric (letting $q = 0$ in each case). Furthermore we see that $\|v+w\| = d(v,-w) \leq d(v,0) + d(0,-w) = d(v,0) + d(-w,0) = \|v\| + \|-w\| = \|v\| + \|w\|$, exploiting the triangle inequality and reflexivity of the metric. I should note, however, that the last step relies on the property $\|v\| = \|-v\|$, a special case of the property that $\|\alpha v\| = |\alpha|\|v\|$ for all $v \in V$ and $\alpha \in \mathbb{R}$, which we will now verify for each proposed norm separately.

(a) $d_2(p,q)$ comes from the Euclidean norm $\|v\| = \sqrt{v_1^2 + \cdots + v_n^2}$, where letting $v = p-q$ gives us the desired metric. In this case we have

$$\|\alpha v\| = \sqrt{\alpha^2 v_1^2 + \cdots + \alpha^2 v_n^2} = \sqrt{\alpha^2}\sqrt{v_1^2 + \cdots + v_n^2} = |\alpha|\|v\|.$$

(b) $d_1(p,q)$ is induced by the norm $\|v\| = |v_1| + \cdots + |v_n|$. By the properties of the absolute value on $\mathbb{R}$, then, we have

$$\|\alpha v\| = |\alpha v_1| + \cdots + |\alpha v_n| = |\alpha||v_1| + \cdots + |\alpha||v_n| = |\alpha|(|v_1| + \cdots + |v_n|) = |alpha|\|v\|.$$

(c) $d_\infty(p,q)$ is induced by the norm $\|v\| = \max\{|v_1|, \ldots, |v_n|\}$. Here we see that

$$\|\alpha v\| = \max\{|\alpha v_1|, \ldots, |\alpha v_n|\} = \max\{|\alpha||v_1|, \ldots, |\alpha||v_n|\} = |\alpha| \max\{|v_1|, \ldots, |v_n|\} = |\alpha|\|v\|.$$

(d) $d_1(f,g)$ is induced by the norm $\|f(x)\| = \int_0^1 |f(x)|dx$, and we have

$$\|\alpha f(x)\| = \int_0^1 |\alpha f(x)|dx = \int_0^1 |\alpha||f(x)|dx = |\alpha| \int_0^1 |f(x)|dx = |\alpha|\|f(x)\|$$

using the fact from basic calculus that $\int cf(x)dx = c\int f(x)dx$ for all $c \in \mathbb{R}$ and all functions $f$.

(e) Finally, let $d_\infty(f,g)$ be induced by the norm $\|f(x)\| = \max_x |f(x)|$. We then see that

$$\|\alpha f(x)\| = \max_x |\alpha f(x)| = \max_x |\alpha||f(x)| = |\alpha| \max_x |f(x)| = |\alpha|\|f(x)\|.$$

Therefore all of the metrics from the previous problem set are indeed induced by norms.

(3) There are several ways to prove the various parts of this problem. The solutions that I present here makes use of an alternate definition of $\|A\|$. First note that

$$\sup_{0\neq v\in\mathbb{R}^n}\frac{|Av|}{|v|}=\inf\{M:|Av|\leq M|v|\ \forall\ v\in\mathbb{R}^n\}.$$

From this we see that we may also write

$$\|A\|=\sup\{|Av|:|v|\leq 1\}$$

as, for any $w$ with $|w|>1$, $\left|\frac{w}{|w|}\right|=1$, so we may scale any vector so that it has norm less than or equal to 1.

Now let $v=v_1e_1+\cdots+v_ne_n$ be any vector in $\mathbb{R}^n$ with $|v|\leq 1$. Then $|v_i|\leq 1$ for all $i=1,\ldots,n$, so that

$$|Av|=|v_1Ae_1+\cdots+v_nAe_n|\leq|v_1||Ae_1|+\cdots+|v_n||Ae_n|\leq|Ae_1|+\cdots+|Ae_n|<\infty.$$

This is true for all $|v|\leq 1$, so in fact $\|A\|=\sup\{|Av|:|v|\leq 1\}\leq|Ae_1|+\cdots+|Ae_n|<\infty$ and $\|A\|$ is well-defined.

To show that this defines a norm on $Mat(n,m)$ we check that $\|A\|$ satisfies the following properties:

- $\|A\|\geq 0$, since for all $v$ such that $|v|\neq 0$, $|Av|\geq 0$ and $|v|>0$, and so $\frac{|Av|}{|v|}\geq 0$.
- If $\|A\|=0$ then $|Av|=0$ for all $|v|\leq 1$, and hence for all $w\in\mathbb{R}^m$ (scaling some $v$ in the closed unit ball appropriately). Therefore $A$ must be the zero matrix. Similarly if $A=0$ then $|Av|=0$ for all $v\in\mathbb{R}^n$, and so $\|A\|=\sup_{v\neq 0}\frac{|Av|}{|v|}=\sup_{v\neq 0}\frac{0}{|v|}=0$.
- Keeping in mind that $(\alpha A)(v)=\alpha(Av)$ for all $A\in Mat(n,m)$, $\alpha\in\mathbb{R}$, and $v\in\mathbb{R}^m$, we see that $\|\alpha A\|=\sup_{|v|\leq 1}|(\alpha A)v|=\sup_{|v|\leq 1}|\alpha||Av|=|\alpha|\sup_{|v|\leq 1}|Av|=|\alpha|\|A\|$.
- Finally,

$$\begin{aligned}\|A+B\|&=\sup_{|v|\leq 1}|(A+B)v|\\&=\sup_{|v|\leq 1}|Av+Bv|\\&\leq\sup_{|v|\leq 1}(|Av|+|Bv|)\\&\leq\sup_{|v|\leq 1}|Av|+\sup_{|v|\leq 1}|Bv|\\&=\|A\|+\|B\|\end{aligned}$$

so the matrix norm satisfies the triangle inequality.

(4) Let $u,v\in B_R\subset V$, so that $\|u\|,\|v\|<R$. Then, for $\lambda\in[0,1]$,

$$\begin{aligned}\|\lambda u+(1-\lambda)v\|&\leq\|\lambda u\|+\|(1-\lambda)v\|\\&=|\lambda|\|u\|+|1-\lambda|\|v\|\\&<|\lambda|R+|1-\lambda|R\\&=R.\end{aligned}$$

Therefore for any $u,v\in B_R$ and $\lambda\in[0,1]$, $\lambda u+(1-\lambda)v\in B_R$, so $B_R$ is convex.

(5) (a) ($\Rightarrow$) Suppose that $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$, and let $C > 0$ be such that $\|v\|_\alpha \leq C\|v\|_\beta$ for all $v \in V$. Then $B_1^\beta = \{v \in V | \|v\|_\beta < 1\}$. Since $\|v\|_\alpha \leq C\|v\|_\beta$, we see that any $v \in B_1^\beta$ also satisfies $\|v\|_\alpha \leq C\|v\|_\beta < C$, so that $v \in B_C^\alpha \Rightarrow B_1^\beta \subset B_C^\alpha$.

($\Leftarrow$) Now suppose that $B_C^\alpha \Rightarrow B_1^\beta \subset B_C^\alpha$ for some $C > 0$. This implies that $\|w\|_\beta \leq C\|w\|_\alpha$ whenever $\|w\|_\beta < 1$.

I will present two ways to proceed from here. The first, and perhaps simplest, approach was suggested to me by Professor Popa. Consider an arbitrary vector $v \in V$. $\|v\|_\beta$ is then simply a scalar; if $v \neq 0$ then $\|v\|_\beta > 0$ and we may rescale $v$ by $\frac{1}{2\|v\|_\beta}$ to obtain a vector $v'$ whose $\beta$-norm is

$$\|v'\|_\beta = \left\|\frac{v}{2\|v\|_\beta}\right\|_\alpha = \frac{1}{2\|v\|_\beta}\|v\|_\beta = \frac{1}{2}$$

so that $v' \in B_1^\beta$. By our hypothesis, then, $v' \in B_C^\alpha$, and so $\|v'\|_\beta \leq \|v'\|_\alpha < C$. Multiplying this equation on both sides by $\|v\|_\alpha$ then gives $\|v\|_\alpha \cdot \|v'\|_\beta = \|v\|_\beta < 2C\|v\|_\alpha$. In the case where $v = 0$ we see that $\|v\|_\alpha = \|v\|_\beta = 2C \cdot 0 = 0$, so in all instances we have $\|v\|_\alpha \leq C'\|v\|_\beta$ for all $v \in V$, where $C' = 2C$.

Alternatively, suppose that there exists some $v \in B_1^\beta$ such that $\|v\|_\alpha > C\|v\|_\beta$ (or equivalently, $\|v\|_\beta < \frac{1}{C}\|v\|_\alpha$). Then $\|\lambda v\|_\beta < 1$ whenever $0 \leq \lambda < \frac{1}{\|v\|_\beta}$. Choose $\lambda$ so that $C\frac{1}{\|v\|_\alpha} < \lambda < \frac{1}{\|v\|_\beta}$; we may choose $\lambda$ in this way so long as $\|v\|_\alpha > C\|v\|_\beta$. Now, however, $\|\lambda v\|_\beta < \frac{1}{\|v\|_\beta} = \|v\|_\beta = 1$, so that $\lambda v \in B_1^\beta$, while $\|\lambda v\|_\alpha > C\frac{1}{\|v\|_\alpha} \cdot \|v\|_\alpha = C \Rightarrow \lambda v \notin B_C^\alpha$, contradicting the hyporthesis of the problem.Therefore $\|v\|_\alpha \leq C\|v\|_\beta$ for all $v$ such that $|v| \leq 1$; scaling these vectors up gives the result for all $v \in \mathbb{R}^n$.

(b) ($\Rightarrow$) Suppose that two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are equivalent. Then by part (a) there exists some $C, C' > 0$ such that $B_1^\beta \subset B_C^\alpha$ and $B_1^\alpha \subset B_{C'}^\beta$. However, multiplying the vectors $v$ in $B_1^\beta$ (resp. $B_1^\alpha$) by $r > 0$ gives precisely the vectors in $B_r^\beta$ (resp. $B_r^\alpha$). The same is true for $B_C^\alpha$ and $B_{C'}^\beta$, so that $B_r^\beta = r \cdot B_1^\beta \subset r \cdot B_C^\alpha = B_{rC}^\alpha$ and $B_r^\alpha = r \cdot B_1^\alpha \subset r \cdot B_{C'}^\beta = B_{rC'}^\beta$. Therefore every open ball of radius $R = rC$ with respect to the $\alpha$ norm contains an open ball of radius $r$ with repect to the $\beta$ norm, and every open ball of radius $R' = rC'$ with respect to the $\beta$ norm contains an open ball of radius $r$ with respect to the *alpha* norm. Lastly, we need to check is that this property holds for open balls centered at *all* points $p \in \mathbb{R}^n$; but noting that $q \in B_r(p) \Leftrightarrow (q-p) \in B_r(0)$ for any norm gives us just that. So equivalent norms define equivalent metrics.

($\Leftarrow$) Now suppose that the metrics defined by the norms $\alpha$ and $\beta$ are equivalent. Then in particular there exsits some $C, C' > 0$ such that $B_1^\beta(0) \subset B_C^\alpha(0)$ and $B_1^\alpha(0) \subset B_{C'}^\beta(0)$, so that by part (a) $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ and $\|\cdot\|_\beta \leq \|\cdot\|_\alpha \Rightarrow \|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are equivalent.

(c) From problem (4)(b) on the last problem set, we know that $d_1, d_2$, and $d_\infty$ are all equivalent, therefore the norms that define them are equivalent by part (b).

(6) (a) Letting $M = \max_i \|e_i\|_\alpha$ and writing $v = v_1e_1 + \cdots + v_ne_n$, we see that

$$\|v\|_\alpha = \|v_1e_1 + \cdots + v_ne_n\|_\alpha \leq |v_1|\|e_1\|_\alpha + \cdots + |v_n|\|e_n\|_\alpha \leq M(|e_1| + \cdots + |e_n|) = M\|v\|_1.$$

(b) Note that we have already shown in problem (5) that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, so that in particular $|v| = \|v\|_2 \leq C\|v\|_1$ for all $v \in \mathbb{R}^n$ for some $C > 0$. Let $\delta = \frac{\varepsilon}{CM}$, where $M$ and $C$ are as before. Then $|v-w| < \delta \Rightarrow \|v-w\|_1 < C|v-w| < \frac{\varepsilon}{M}$, and by the triangle inequality we have

$$\begin{aligned} |\|v\|_\alpha - \|w\|_\alpha| &\leq \|v-w\|_\alpha \\ &< M\|v-w\|_1 \\ &< M\frac{\varepsilon}{M} \\ &= \varepsilon \end{aligned}$$

Thus, for every $\varepsilon > 0$ there exists a $\delta > 0$ so that $|\|v\|_\alpha - \|w\|_\alpha| < \varepsilon$ whenever $|v-w| < \delta$, and so the function $f(v) = \|v\|_\alpha$ is continuous, as desired.

(c) Because $S$ is compact and $\|\cdot\|_\alpha$ is continuous, $\|\cdot\|_\alpha$ attains its minimum (and maximum) value on $S$. Let $m$ be this minimum. Note that $m \neq 0$, because $m = 0 \Leftrightarrow \|w\|_\alpha = 0$ for some $w \in S \Leftrightarrow w = 0$. But if $w \in S$ then $|w| = 1$, so $w$ cannot be 0 and thus $m > 0$. Now, by definition of a minimal element we know that $0 \neq m \leq \|w\|_\alpha$ for all $w$ with $\|w\|_1 = 1$; multiplying both sides of this inequality by $\|w\|_1 = 1$ gives

$$m\|w\|_\alpha \leq \|w\|_\beta \Rightarrow \|w\|_1 \leq \frac{1}{m}\|w\|_\alpha.$$

Let $v$ be any vector in $\mathbb{R}^n$. If $v = 0$ then $\|v\|_1 = \frac{1}{m}\|v\|_\alpha = 0$. Otherwise the vector $w = \frac{v}{\|v\|_1}$ has the property $\|w\|_1 = 1$, so that by our previous work

$$\left\|\frac{v}{\|v\|_1}\right\|_1 \leq \frac{1}{m}\left\|\frac{v}{\|v\|_1}\right\|_\alpha.$$

Multiplying both sides of the equation by the scalar $\|v\|_1$ then gives

$$\|v\|_1 \leq \frac{1}{m}\|v\|_\alpha$$

for all $v \in \mathbb{R}^n$. This completes the proof: since all norms are equivalent to $\|\cdot\|_1$ the **amazing** fact follows that all norms are equivalent to one another.

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