

PROBLEM SET #4 SOLUTIONS  
PART A  
October 25, 2002

(1) (Problem 1.7.8)

(a) This problem asks you to recall the definition of the Jacobian matrix. In the notation of the book, the  $i, j$ th entry of  $\mathbf{J}f$  evaluated at  $a$  is  $D_i f_j(a)$ . From this we see that

$$D_1 f_1 = 2x \cos(x^2 + y), \quad D_2 f_1 = \cos(x^2 + y), \quad \text{and} \quad D_2 f_2 = x e^{xy}.$$

(b) Since  $f$  maps  $\mathbb{R}^2$  to  $\mathbb{R}^3$ ,  $\mathbf{J}f$  is a  $3 \times 2$  matrix.

(2) (Problem 1.7.21) Recall that for  $2 \times 2$  matrices, the determinant is the function mapping  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto ad - bc$ . Equivalently we can think of the determinant as mapping the vector  $(a, b, c, d)^t \in \mathbb{R}^4$  to  $ad - bc$ , and similarly we will identify the increment matrix  $H$  with  $(h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2})^t$ . Then we have

$$\mathbf{J} \det(a_{i,j}) = [a_{2,2} \quad -a_{2,1} \quad -a_{1,2} \quad a_{1,1}]$$

Evaluating this at  $I$  (identified with  $(1, 0, 0, 1)^t$ ), we have

$$\mathbf{J} \det(I) = [1 \quad 0 \quad 0 \quad 1]$$

so that

$$[\mathbf{D} \det(I)]H = [1 \quad 0 \quad 0 \quad 1] \begin{bmatrix} h_{1,1} \\ h_{1,2} \\ h_{2,1} \\ h_{2,2} \end{bmatrix} = h_{1,1} + h_{2,2}$$

as desired.

(3) (Problem 1.9.2 (a)) We first note that, away from the origin, the function  $f$  is a quotient of polynomials with denominator never equal to zero; therefore  $f$  is continuous and differentiable, and all directional derivatives exist at every point. Applying the definition of a directional derivative at the origin, we see that the derivative of  $f$  in the direction of  $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  is

$$\begin{aligned} \mathbf{D}_v f(0) &= \lim_{h \rightarrow 0} \frac{f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + h \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \frac{3h^3 v_1^2 v_2 - v_2^3}{h^2(v_1^2 + v_2^2)} \\ &= \frac{3v_1^2 v_2 - v_2^3}{v_1^2 + v_2^2}. \end{aligned}$$

For  $v \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , this quantity is well-defined, and so that all directional derivatives exist here as well.

However, if  $f$  is differentiable at the origin, then as in Example 1.9.3 in the text the directional derivative of  $f$  at 0 in the direction of an arbitrary vector  $v$  is given by

$$\mathbf{D}_v f(0) = [\mathbf{J}f(0)]v.$$

In this case the Jacobian of  $f$  is given by

$$\begin{aligned}\frac{\partial f}{\partial x}\Big|_{0,0} &= \lim_{h \rightarrow 0} \frac{f\begin{bmatrix} 0+h \\ 0 \end{bmatrix} - f\begin{bmatrix} 0 \\ 0 \end{bmatrix}}{h} = 0 \\ \frac{\partial f}{\partial y}\Big|_{0,0} &= \lim_{h \rightarrow 0} \frac{f\begin{bmatrix} 0 \\ 0+h \end{bmatrix} - f\begin{bmatrix} 0 \\ 0 \end{bmatrix}}{h} = -1 \\ &\Rightarrow \mathbf{J}f(0) = [0 \quad -1]\end{aligned}$$

From which the directional derivative should be

$$\begin{aligned}\mathbf{J}f(0,0) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= -v_2 \\ &\neq \mathbf{D}_v f(0).\end{aligned}$$

Therefore  $f$  is not differentiable at the origin.

- (4) (a) Following the hint, we will compose  $f$  with the map  $g$  sending the polar coordinates  $\begin{pmatrix} r \\ \theta \end{pmatrix}$  to their corresponding Cartesian coordinates  $\begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$ . Since  $f$  and  $g$  are both differentiable, so is the  $g \circ f$ , and the Jacobian matrix of this function is given by the chain rule to be

$$\begin{aligned}\mathbf{J}(g \circ f)(r, \theta) &= \mathbf{J}f(g(r, \theta)) \cdot \mathbf{J}g(r, \theta) \\ &= \mathbf{J}f(x, y) \cdot \mathbf{J}g(r, \theta) \\ &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta & -r \frac{\partial f}{\partial x} \sin \theta + r \frac{\partial f}{\partial y} \cos \theta \end{bmatrix}.\end{aligned}$$

- (b) (Problem 1.8.6 (a)) I will prove this in full generality. Recalling the definition of the derivative of  $f \cdot g$  at  $a$ , we have

$$[\mathbf{D}(f \cdot g)(a)]v = \lim_{h \rightarrow 0} \frac{(f \cdot g)(a + hv) - (f \cdot g)(a)}{h}$$

Keeping in mind that the dot product is bilinear (so that  $(u + v) \cdot w = u \cdot w + v \cdot w$ ,  $u \cdot (v + w) = u \cdot v + u \cdot w$ , and  $\alpha(u \cdot v) = (\alpha u) \cdot v = u \cdot (\alpha v)$ ), we have

$$\begin{aligned}[\mathbf{D}(f \cdot g)(a)]v &= \lim_{h \rightarrow 0} \frac{f(a + hv) \cdot g(a + hv) - f(a) \cdot g(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a + hv) \cdot g(a + hv) - f(a) \cdot g(a + hv) + f(a) \cdot g(a + hv) - f(a) \cdot g(a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(f(a + hv) - f(a)) \cdot g(a + hv) + f(a) \cdot (g(a + hv) - g(a))}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(a + hv) - f(a)}{h} \cdot g(a + hv) + \lim_{h \rightarrow 0} f(a) \cdot \frac{g(a + hv) - g(a)}{h} \\ &= [\mathbf{D}f(a)]v \cdot g(a) + f(a) \cdot [\mathbf{D}g(a)]v\end{aligned}$$

as desired.

- (5) Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}^3$  be given by  $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix}$  and  $g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix}$ . Then the cross product  $f \times g : \mathbb{R} \rightarrow \mathbb{R}^3$  is given by

$$(f \times g)(x) = \begin{pmatrix} f_2(x)g_3(x) - f_3(x)g_2(x) \\ -f_1(x)g_3(x) + f_3(x)g_1(x) \\ f_1(x)g_2(x) - f_2(x)g_1(x) \end{pmatrix}.$$

Making liberal use of the single-variable product rule, we find the Jacobian of this function to be

$$\begin{aligned} \mathbf{J}(f \times g) &= \begin{bmatrix} \frac{\partial f_2}{\partial x} g_3 + f_2 \frac{\partial g_3}{\partial x} - \frac{\partial f_3}{\partial x} g_2 - f_3 \frac{\partial g_2}{\partial x} \\ -\frac{\partial f_1}{\partial x} g_3 - f_1 \frac{\partial g_3}{\partial x} + \frac{\partial f_3}{\partial x} g_1 + f_3 \frac{\partial g_1}{\partial x} \\ \frac{\partial f_1}{\partial x} g_2 + f_1 \frac{\partial g_2}{\partial x} - \frac{\partial f_2}{\partial x} g_1 - f_2 \frac{\partial g_1}{\partial x} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f_2}{\partial x} g_3 - \frac{\partial f_3}{\partial x} g_2 \\ -\frac{\partial f_1}{\partial x} g_3 + \frac{\partial f_3}{\partial x} g_1 \\ \frac{\partial f_1}{\partial x} g_2 - \frac{\partial f_2}{\partial x} g_1 \end{bmatrix} + \begin{bmatrix} f_2 \frac{\partial g_3}{\partial x} - f_3 \frac{\partial g_2}{\partial x} \\ -f_1 \frac{\partial g_3}{\partial x} + f_3 \frac{\partial g_1}{\partial x} \\ f_1 \frac{\partial g_2}{\partial x} - f_2 \frac{\partial g_1}{\partial x} \end{bmatrix} \\ &= \mathbf{D}f \times g + f \times \mathbf{D}g \end{aligned}$$

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