

# Math 25a Homework 4 Part B Solutions

by Luke Gustafson

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**1.** We have  $\sum_{i=1}^n x_i D_i f(x) = (D_1 f(x) D_2 f(x) \dots D_n f(x)) \cdot (x_1 \ x_2 \ \dots \ x_n) = x \cdot Df(x) = D_x f(x)$ , the directional derivative of  $f$  in the direction  $x$  at the point  $x$ . By definition of the directional derivative and homogeneity,  $D_x f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(1+h)^m f(x) - f(x)}{h} = f(x) \lim_{h \rightarrow 0} \frac{(1+h)^m - 1}{h}$ . Observe that  $\lim_{h \rightarrow 0} \frac{(1+h)^m - 1}{h}$  is the derivative of  $x^m$  at  $x = 1$ , so this limit is  $m$ . Hence, we find  $D_x f(x) = m f(x)$ .

**2a.** (Note that the  $<$  sign should be  $\leq$  in the definition of Lipschitz, otherwise it couldn't hold for  $x = y$ .) Fix the  $M', \delta'$  given by the local Lipschitz condition at  $x$ . Given  $\epsilon > 0$ , take  $\delta = \min\{\delta', \epsilon/M'\}$ . Then, whenever  $|x - y| < \delta$ ,  $|x - y| < \delta'$  so we may use the Lipschitz condition. It gives us  $|f(x) - f(y)| \leq M'|x - y| < M'\delta \leq M'\epsilon/M' = \epsilon$ , which proves that  $f$  is continuous at  $x$ . This holds for all  $x$ , so  $f$  is continuous.

**2b.** At a given point  $x$ , we have  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - Df(x)(h)}{|h|} = 0$ . Let  $y = x + h$ , so  $h \rightarrow 0$  if and only if  $y \rightarrow x$ . Thus  $\lim_{y \rightarrow x} \frac{f(y) - f(x) - Df(x)(y-x)}{|y-x|} = 0$ . By definition of a limit, for  $\epsilon = 1$  there is a  $\delta$  so that  $\left| \frac{f(y) - f(x) - Df(x)(y-x)}{y-x} \right| < 1$  for  $|x - y| < \delta$ ,  $x \neq y$ . By the triangle inequality,  $\left| \frac{f(y) - f(x)}{y-x} \right| - \left| \frac{Df(x)(y-x)}{y-x} \right| \leq \left| \frac{f(y) - f(x) - Df(x)(y-x)}{y-x} \right|$ . Thus,  $\left| \frac{f(y) - f(x)}{y-x} \right| - \left| \frac{Df(x)(y-x)}{y-x} \right| < 1$  so  $|f(y) - f(x)| < \left( \left| \frac{Df(x)(y-x)}{y-x} \right| + 1 \right) |y - x|$ . By previous homework,  $\left| \frac{Df(x)(y-x)}{y-x} \right|$  has a maximum value  $c$ . Taking  $M = 1 + c$  gives us the Lipschitz condition, so a differentiable function is locally Lipschitz.

**2c.**  $f(x) = x^{1/3}$  is continuous. However, at  $x = 0$ , if the function were Lipschitz, we would have  $|y^{1/3} - 0| \leq M|y - 0|$  whenever  $|y - 0| < \delta$ . This is false for  $|y| < \min\{M^{-3/2}, \delta\}$ , as can easily be seen by plugging in the values.

$f(x) = |x|$  is not differentiable at 0. However, this function is Lipschitz. Use the triangle inequality to show  $-|x - y| \leq |x| - |y| \leq |x - y|$ , which implies  $||x| - |y|| \leq |x - y|$ . This proves  $f$  is Lipschitz.

**3.** First, we show the second generalization is false. Take  $f(x) = (x^2 \ x^3)$ ,  $a = -b \neq 0$ . Then,  $Df(c) = (2c \ 3c^2)$ . Plugging in these values, the components of our equation become  $0 = \lambda(2c)(2b)$  and  $2b^3 = \lambda(3c^2)(2b)$ . Since the first equation is zero,  $\lambda$  or  $c$  is zero. However, then  $\lambda(3c^2)(2b) = 0$ , contradicting  $2b^3 \neq 0$ . Then, the second generalization can't always be true.

For  $m = 2$ ,  $n = 1$ , the first generalization is the second with  $\lambda = 1$ , so the previous example shows it does not hold. For other  $m$ , take  $f(x) = (x^2 \ x^3 \ 0 \ 0 \ \dots)$ , and an identical argument shows it does not hold.

*Remark:* In the spirit of the hint, one could imagine a particle moving in the circle  $(\sin x \ \cos x)$ . It is in the same position at  $x = 0$  and  $x = 2\pi$ , but its velocity, and hence its derivative, is never 0, which gives a contradiction to the first generalization.

*Remark 2:* Several people found that the extended mean value theorem almost implies the second generalization is true. (Roughly speaking, if you divide the equation for one component by the equation for the other, you get the expression in the extended mean value theorem.) However, to use the extended mean value theorem, it requires that if  $f = (f_1 \ f_2)$ , then either  $f'_1$  or  $f'_2$  is nonzero in  $(a, b)$ . Note that this is not true for our counterexample.