# Math 25a Homework 5 Part A Solutions

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**1.** Consider two rows $R$ and $S$. Using only the given two row operations, we can exchange the rows as follows: $\begin{pmatrix} R \\ S \end{pmatrix} \to \begin{pmatrix} R \\ S-R \end{pmatrix} \to \begin{pmatrix} S \\ R-S \end{pmatrix} \to \begin{pmatrix} S \\ R \end{pmatrix}$.

**2.** Use row reduction. We begin with the matrix

$$\begin{bmatrix} 1 & -1 & -1 & -3 & 1 & 1 \\ 1 & 1 & -5 & -1 & 7 & 2 \\ -1 & 2 & 2 & 2 & 1 & 0 \\ -2 & 5 & -4 & 9 & 7 & \beta \end{bmatrix}$$

and row reducing gives

$$\begin{bmatrix} 1 & 0 & 0 & -4 & 3 & 2 \\ 0 & 1 & 0 & -1/3 & 7/3 & 5/6 \\ 0 & 0 & 1 & -2/3 & -1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1+2\beta \end{bmatrix}$$

(You were expected to show the work of row reducing, but it is omitted here because I wrote on your papers where you made mistakes if you didn't get it right.) For this system to have solutions, we must have $1 + 2\beta = 0$, so $\beta = -1/2$. Then, we can solve for the $x$ in terms of two independently chosen variables as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 + 4\alpha - 3\beta \\ 5/6 + 1/3\alpha - 7/3\beta \\ 1/6 + 2/3\alpha + 1/3\beta \\ \alpha \\ \beta \end{pmatrix}$$

**3.** Use row reduction. We begin with the matrix

$$\begin{bmatrix} 1 & 1 & a & 1 \\ 1 & a & 1 & 1 \\ a & 1 & 1 & a \end{bmatrix}$$

and row reducing (watch out for division by 0!) gives

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

when $a \neq -2, 1$ (in these cases, there would be division by zero when row reducing). (Again, you were expected to show the work, but I corrected the work you submitted, so the row reduction is omitted here.) In this case, we have $x = 1, y = 0, z = 0$ as the unique solution.

When $a = 1$, all three equations become $x + y + z = 1$. So, we can pick $y$ and $z$ arbitrarily and let $x = 1 - y - z$. Thus, there are infinitely many solutions.

When $a = -2$, we substitute for $a$ and then row reduce to get

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, we can pick $z$ arbitrarily and set $y = z$ and $x = 1 + z$. Thus, there are infinitely many solutions.

**4.** Again, since I pointed out where mistakes were made in the row reducing, those steps are omitted here and just the answers are given. If $A$ is the given matrix, row reducing $(A|I)$ gives the following:

**a.** $\begin{bmatrix} 1 & 0 & 1/6 & 5/54 \\ 0 & 1 & -1/6 & 1/54 \end{bmatrix}$, and the inverse exists and is $\begin{bmatrix} 1/6 & 5/54 \\ -1/6 & 1/54 \end{bmatrix}$

**b.** $\begin{bmatrix} 1 & 3 & 0 & 1/3 \\ 0 & 0 & 1 & -1/3 \end{bmatrix}$. The left half does not reduce to the identity, so the matrix has no inverse.

**c.** $\begin{bmatrix} 1 & 0 & 0 & 3/2 & -1/4 & -9/4 \\ 0 & 1 & 0 & -1 & 1/2 & 3/2 \\ 0 & 0 & 1 & 1/2 & -1/4 & -1/4 \end{bmatrix}$, and the inverse exists and is the right half of this matrix.

**d.** The inverse does not exist because this matrix is not square.

**e.** $\begin{bmatrix} 1 & 0 & 0 & 1/7 & -1/2 & 1/14 \\ 0 & 1 & 0 & 4/21 & 5/6 & -1/14 \\ 0 & 0 & 1 & -4/21 & 1/6 & 1/14 \end{bmatrix}$, and the inverse exists and is the right half of this matrix.

**f.** $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$, and the inverse exists and is the right half of this matrix.

**g.** $\begin{bmatrix} 1 & 0 & 0 & 0 & 4 & -6 & 1 & -1 \\ 0 & 1 & 0 & 0 & -6 & 14 & -11 & 3 \\ 0 & 0 & 1 & 0 & 4 & -11 & 10 & -3 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{bmatrix}$, and the inverse exists and is the right half of this matrix.

**5a.** Row reducing gives

$$A' = \begin{bmatrix} 1 & 0 & 0 & 3a - b - 4c \\ 0 & 1 & 0 & a - b - c \\ 0 & 0 & 1 & -2a + b + 3c \end{bmatrix}$$

**5b.** The inverse is

$$B^{-1} = \begin{bmatrix} 3 & -1 & 4 \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$$

**5c.** The matrix $B^{-1}$ consists of the coefficients of $a$, $b$, and $c$ in the row-reduced matrix $A'$. Let $A = (B|x)$, where $x$ is the vector $(a\ b\ c)$. Then, row-reducing $A$ is equivalent to applying a matrix $E$, the product of the matrices representing the row operations. Looking at the left three columns of $A' = (EB|Ex)$, we know $EB = I$, so $E = B^{-1}$. Then, the rightmost column is $Ex = B^{-1}x$, so $B^{-1}$ must be the coefficients of $a$, $b$, and $c$, as expected.