Problem Set #5 Solutions
Part B
November 1, 2002

"Let's play with this gadget and see what happens."

–Barry Mazur

(1) (Problem 2.3.5)

(a) Form the augmented matrix $\left[\begin{smallmatrix}3&-1&3&1\\2&1&-2&1\\1&1&1&1\end{smallmatrix}\right]$ and row reduce. One possible sequence of steps in this reduction is given below.

$$\begin{bmatrix}3&-1&3&1\\2&1&-2&1\\1&1&1&1\end{bmatrix}\to\begin{bmatrix}1&-2&5&0\\2&1&-2&1\\1&1&1&1\end{bmatrix}\to\begin{bmatrix}1&-2&5&0\\0&-1&-4&-1\\1&1&1&1\end{bmatrix}\to$$

$$\begin{bmatrix}1&-2&5&0\\0&-1&-4&-1\\0&3&-4&1\end{bmatrix}\to\begin{bmatrix}1&0&13&2\\0&-1&-4&-1\\0&3&-4&1\end{bmatrix}\to\begin{bmatrix}1&0&13&2\\0&-1&-4&-1\\0&0&-16&-2\end{bmatrix}\to$$

$$\begin{bmatrix}1&0&13&2\\0&1&4&1\\0&0&8&1\end{bmatrix}\to\begin{bmatrix}1&0&0&\frac{3}{8}\\0&1&4&1\\0&0&8&1\end{bmatrix}\to\begin{bmatrix}1&0&0&\frac{3}{8}\\0&1&0&\frac{1}{2}\\0&0&1&\frac{1}{8}\end{bmatrix}$$

$$\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{3}{8}\\\frac{1}{2}\\\frac{1}{8}\end{pmatrix}$$

is the solution to this system.

(b) We can also find the inverse of the matrix $\left[\begin{smallmatrix}3&-1&3\\2&1&-2\\1&1&1\end{smallmatrix}\right]$ by forming the augmented matrix $\left[\begin{smallmatrix}3&-1&3&1&0&0\\2&1&-2&0&1&0\\1&1&1&0&0&1\end{smallmatrix}\right]$ and row reducing:

$$\begin{bmatrix}3&-1&3&1&0&0\\2&1&-2&0&1&0\\1&1&1&0&0&1\end{bmatrix}\to\begin{bmatrix}1&-2&5&1&-1&0\\2&1&-2&0&1&0\\1&1&1&0&0&1\end{bmatrix}\to\begin{bmatrix}1&-2&5&1&-1&0\\0&-1&-4&0&1&-2\\1&1&1&0&0&1\end{bmatrix}\to$$

$$\begin{bmatrix}1&-2&5&1&-1&0\\0&-1&-4&0&1&-2\\0&3&-4&-1&1&1\end{bmatrix}\to\begin{bmatrix}1&-2&5&1&-1&0\\0&-1&-4&0&1&-2\\0&0&-16&-1&4&-5\end{bmatrix}\to\begin{bmatrix}1&-2&5&1&-1&0\\0&1&0&-\frac{1}{4}&0&\frac{3}{4}\\0&0&-16&-1&4&-5\end{bmatrix}\to$$

$$\begin{bmatrix}1&0&5&\frac{1}{2}&-1&\frac{3}{2}\\0&1&0&-\frac{1}{4}&0&\frac{3}{4}\\0&0&-16&-1&4&-5\end{bmatrix}\to\begin{bmatrix}1&0&5&\frac{1}{2}&-1&\frac{3}{2}\\0&1&0&-\frac{1}{4}&0&\frac{3}{4}\\0&0&1&\frac{1}{16}&-\frac{1}{4}&\frac{5}{16}\end{bmatrix}\to\begin{bmatrix}1&0&0&\frac{3}{16}&\frac{1}{4}&-\frac{1}{16}\\0&1&0&-\frac{1}{4}&0&\frac{3}{4}\\0&0&1&\frac{1}{16}&-\frac{1}{4}&\frac{5}{16}\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}3&-1&3\\2&1&-2\\1&1&1\end{bmatrix}^{-1}=\begin{bmatrix}\frac{3}{16}&\frac{1}{4}&-\frac{1}{16}\\-\frac{1}{4}&0&\frac{3}{4}\\\frac{1}{16}&-\frac{1}{4}&\frac{5}{16}\end{bmatrix}$$

From which we find

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{1}{2} \\ \frac{1}{8} \end{pmatrix}$$

as expected.

(2) (Problem 2.3.7)

(a) $AA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) Since $A = A^{-1}$, we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} 5 \\ 7 \\ 11 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -9 \end{pmatrix}.$$

(3) Consider the product matrix AB whose (i, j)th element is given by

$$AB_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$

If A and B are both upper triangular, then $A_{ij} = B_{ij} = 0$ whenever $i > j$, so that $AB_{ij} = \sum_{r=1}^{i-1} A_{ir} B_{rj} + \sum_{r'=i}^{n} A_{ir'} B_{r'j}$. Now, for $i > j$ we have that $i > r$ in the first summation, so that all of the terms A_{ir} are zero, while in the second $r' \geq i > j$, so that all of the $B_{r'j}$ are zero. Thus $AB_{ij} = 0 \Rightarrow AB$ is upper triangular.

Similarly if A and B are both lower triangular, then for $i < j$ the term $AB_{ij} = \sum_{r=1}^{j-1} A_{ir} B_{rj} + \sum_{r'=j}^{n} A_{ir'} B_{r'j} = 0 + 0 = 0$, so that AB is also lower triangular.

(4) Note that each elementary row operation is given by a matrix. Consider the following remarks concerning these matrices, which I will leave to the reader to verify:

- The product of transposition (row-exchange) matrices is a permutation matrix.
- The matrices corresponding to multiplication of a row by a scalar are upper triangular and lower triangular, and the matrices corresponding to addition of a scalar multiple of one row to another are either upper or lower triangular.
- The matrices corresponding to elementary row operations are invertible, and their inverses are upper/lower triangular whenever the original matrices are so.
- It is possible to write the product BE of two elementary transformations as $E'B'$, where E' gives the same type of elementary row operation as E, and B' gives the same type of elementary row operation as B.

We now note that it is possible to reduce any matrix A via elementary row operations to its reduced row-echelon form A', and that A' is upper triangular in the sense that $A'_{ij} = 0$ whenever $i > j$. Thus we have

$$A' = E_k \cdots E_1 A$$

where each of the E_i is the matrix of an elementary row operation. Making use of the above remarks, we may rewrite this as $A' = U_l \cdots U_1 L_m \cdots L_1 T_n \cdots T_1 A$, where the U_i are upper triangular matrices corresponding to addition of one row to another or multiplication of a row by a scalar, the L_j are lower triangular matrices corresponding to the same operations, and the T_k are transposition matrices. We may then multiply both sides of this equation by the inverses of the U_i and L_j to obtain

$$\begin{aligned} T_n \cdots T_1 A = PA &= L_1^{-1} \cdots L_m^{-1} U_1^{-1} \cdots U_l^{-1} A' \\ &= LU \end{aligned}$$

as desired.

(5) We know that for any matrix A there exists a set of (invertible) matrices, corresponding to elementary row operations, such that $E_k \cdots E_1 A = PA$ is in reduced row-echelon form. We may analogously define column-echelon form (as row-echelon form of A^t), and obtain it by multiplying on the right by elementary matrices $F_1 \cdots F_l = Q$, which is also invertible.

Claim. *PAQ, with P and Q given as before, is in row-echelon as well as column-echelon form.*

Proof. Consider in turn each elementary column operation. Since PA is already in row-echelon form, we only need to subtract columns from columns lying to the right (to reduce non-pivotal columns to pivotal ones), or move zero columns to the right by column exchange. In the first case the row-echelon property is preserved because no pivotal column is made into a nonpivotal column, and the matrix remains upper-triangular. It is clear in the second case that the matrix remains in row-echelon form. Thus reducing PA to column-echelon form does not change it from row-echelon form, thus PAQ is in both row-echelon and column-echelon form. □

Now examine the form of PAQ. Since PAQ is in row-echelon form, the first nonzero element of every row must be a 1, and no other row may have a nonzero entry in this position. Since PAQ is in column-echelon form, the first nonzero element of every column must also be a 1, and no other column may have a nonzero entry in this position. Furthermore we have arranged the rows so that the top r of them have nonzero entries, while the rest are zero rows, and we have arranged the columns so that the leftmost s of them have nonzero entries, with the rest being zero rows. Thus if the i, jth entry of PAQ is 1, then there are no other nonzero entries in either the ith row or the jth column, and we must have $i = j$, because otherwise PAQ would not be in row-echelon form, or it would not be in column-echelon form. Hence $r = s$ and PAQ takes the desired form $\left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

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