Problem Set #6 Solutions
Part A
November 8, 2002

(1) Consider the set $\{e_{ij}\}$, $1 \le i \le n$ and $1 \le j \le m$, where e_{ij} is the $n \times m$ matrix with a 1 in the (i, j) entry and zeros everywhere else. These matrices are linearly independent, for in order for a linear combination $\sum_{i,j} c_{ij} e_{ij}$ to equal the zero matrix we must have the (i, j) entry equal zero for all i, j. Therefore all of the $c_{ij} = 0$ as well. Furthermore this set spans $Mat(n, m)$, since every $A \in Mat(n, m)$ is specified by its elements a_{ij}, so that clearly $A = \sum_{i,j} a_{ij} e_{ij}$. Thus the e_{ij} form a basis for $Mat(n, m)$.

We can in fact generalize this notion in the following way to find a basis for the vector space of linear transformations from V to W. Let $T_{ij} \in Lin(V, W)$ $(1 \le i \le n, 1 \le j \le m)$ be the linear map that sends the basis vector v_i to the basis vector w_j and all other basis vectors of V to zero (actually, to have things be completely analogous to the matrix case we should let T_{ji} be the map sending v_i to w_j, but this works just as well). Again, the T_{ij} are linearly independent, for if the linear transformation $S = \sum_{i,j} c_{ij} T_{ij} = 0$ then in particular $S(v_{i'}) = 0$ for all (fixed) i'. But $S(v_{i'}) = \sum_j c_{i'j} w_j$; since the w_j are linearly independent we must have that $c_{i'j} = 0$ for all j. Running through all i' then gives us $c_{ij} = 0$ for all i, j.

Furthermore the T_{ij} span $Lin(V, W)$, which we show as follows. Recall that every linear transformation $L : V \to W$ is uniquely determined by how it acts on a basis for V. Suppose that, for each i, $L(v_i) = \sum_j a_{ij} w_j$.

Claim. $L = \sum_{i,j} a_{ij} T_{ij}$.

Proof. Since the v_i are a basis for V, we need only show that $L(v_i) = (\sum_{i',j} a_{i'j} T_{i'j})(v_i)$ for all i. By definition $T_{i'j}(v_i) = 0$ whenever $i \ne i'$, so we see that

$$\left(\sum_{i',j} a_{i'j} T_{i'j} \right) (v_i) = \sum_j a_{ij} T_{ij}(v_i) = \sum_j a_{ij} w_j = L(v_i)$$

as desired. □

Therefore every linear transformation L can be written as a linear combination of the T_{ij}, so that these transformations span $\Rightarrow$ the T_{ij} are a basis for $Lin(V, W)$.

(2) (Problem 2.4.12(a)) We will show that the matrix $\tilde{A}$ satisfies all of the components of the definition given on p. 173 of the text. To see that the first nonzero entry of every row is a 1, note that the marked columns $\vec{a}_{i_j}$ are given by $\vec{e}_j$ in $\tilde{A}$. Then all columns before $\vec{a}_{i_j}$ are linear combinations of the earlier columns, and hence have the form $\tilde{\vec{a}}_l$, which has all zeros below the $(j-1)$th entry. So the first entry in the jth row is 1, for all j.

The pivotal 1's are all located in the marked columns of $\tilde{A}$, and by construction each marked column $\vec{a}_{i_j}$ is given by $\vec{e}_j$ in $\tilde{A}$. Now, we have that the indices of the marked columns are $i_1 < \cdots < i_k$, so that the $\vec{e}_j$ appear in order, so that the pivotal 1 of a lower (index) row always appears to the right of the pivotal 1 of a higher row.

Every column containing a pivotal 1 has all other entries equal to zero, because the pivotal 1's are located exactly in the columns $\tilde{\vec{a}}_{i_j} = \vec{e}_j$.

Finally, we see that every row above and including row k, where i_k is the highest index of the marked columns $\vec{a}_{i_j}$ must have one nonzero entry, for the first nonzero entry of every row is 1, and the highest index row for which we have a nonzero entry is k. Thus for every row below the kth, all entries are zero.

Hence $\tilde{A}$ is in row echelon form.

(3) ($\Rightarrow$) Suppose that $\{v_1, \ldots, v_n\}$ is a basis for V, and let L be the linear map sending v_i to w_i for some vectors $w_1, \ldots, w_n \in W$. Suppose that L' is another map with this same property, and consider the map $L - L'$. Since the set $\{v_i\}$ forms a basis for V, every $v \in V$ can be uniquely written as $\sum_{i=1}^n c_i v_i$. From this we see that $(L - L')(v) = (L - L')(\sum_{i=1}^n c_i v_i) = \sum_{i=1}^n c_i[L(v_i) - L'(v_i)] = \sum_{i=1}^n c_1[w_i - w_i] = 0$; since this holds for all $v \in V$ we have $(L - L') = 0 \Rightarrow L = L'$, so that L is unique.

($\Leftarrow$) Now suppose that there exists a unique map L that acts on a set of vectors $\{v_i\}$ in the same way as above. If the v_i are not linearly independent, then there exist constants a_i, $1 \leq i \leq n-1$ such that (without loss of generality) $v_n = \sum_{i=1}^{n-1} a_i v_i$. Choose a set of vectors $w_1, \ldots, w_{n-1}, w'_n \in W$ such that $w'_n \neq \sum_{i=1}^{n-1} a_i w_i$. Such a set of vectors can always be found, for every field has at least the elements 1 and 0, so we may always let the w_i be zero vectors and w'_n be nonzero. Then we have a (unique) linear map L sending v_i to w_i for $1 \leq i \leq n-1$, and sending v_n to w'_n. But if this is the case then $L(v_n) = w'_n = L(\sum_{i=1}^{n-1} a_i v_i) = \sum_{i=1}^{n-1} a_i w_i \neq w'_n$, a contradiction. So the v_i are linearly independent.

If the v_i do not span, then there exists $u_{n+1}, \ldots, u_p$ such that the set $\{v_1, \ldots, v_n, u_{n+1}, \ldots, u_p\}$ is a basis for V. Since a linear transformation is uniquely determined by what it does to a basis, we may consider the map L sending v_i to w_i for $1 \leq i \leq n$ and u_j to w_j for some $w_j \in W$, and the map L' sending v_i to w_i for $1 \leq i \leq n$ and u_j to $w'_j \neq w_j$ for $n+1 \leq j \leq p$. Clearly $L - L' \neq 0 \Rightarrow L \neq L'$, for $(L - L')(u_{n+1}) = w_{n+1} - w'_{n+1} \neq 0$, but both L and L' send v_i to w_i for all $1 \leq i \leq n$. Hence L is not unique, again a contradiction. So the v_i span $V \Rightarrow$ the v_i form a basis for V.

(4) We first note that, because we are dealing with a finite-dimensional vector space, every subspace (and in particular V, W, $V \cap W$, and $V + W$) is finite-dimensional, and therefore has a basis with a finite number of elements. Let $u_1, \ldots u_k$ be a basis for $V \cap W$. Then the u_i are linearly independent in both V and W, hence can be extended to bases $\{u_1, \ldots, u_k, v_{k+1}, \ldots, v_n\}$ for V and $\{u_1, \ldots, u_k, w_{k+1}, \ldots, w_m\}$ for W. Now, every basis for a vector space has the same number of elements, so we may simply consider the number of elements in the bases we have chosen in order to determine the dimensions of the various subspaces we are considering. We have that $\{u_1, \ldots, u_k, v_{k+1}, \ldots, v_n, w_{k+1}, \ldots, w_m\}$ is a basis for $V + W$ (you should verify this), so that $\dim(V + W) = n + m - k$. On the other hand $\dim V = n$, $\dim W = m$, and $\dim(V \cap W) = k$, so that $\dim(V + W) = \dim V + \dim W - \dim(V \cap W)$, as desired.

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