Math 25a Homework 6 Part B Solutions

by Luke Gustafson

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1a. If the vectors were dependent, then we would have a nontrivial relation $a_1w_1 + \cdots a_i\frac{1}{\alpha}w_i + \cdots + a_mw_m = 0$ of the vectors $w_1, \ldots, \frac{1}{\alpha}w_i, \ldots, w_m$. Then, $a_1w_1 + \cdots \frac{a_i}{\alpha}w_i + \cdots + a_mw_m = 0$ is a relation of the vectors $w_1, \ldots, w_i, \ldots, w_m$. Therefore, $w_1, \ldots, \frac{1}{\alpha}w_i, \ldots, w_m$ is linearly independent if $a_1w_1 + \cdots w_i + \cdots + a_mw_m = 0$ is.

Any set of m linearly independent vectors is a basis of an m-dimensional vector space, so the new set of vectors must also be a basis.

To convert a vector $(w_1\ w_2\ \ldots)$ from the original basis to the new basis, we just multiply w_i by α. The matrix that does this is $B = E_1(i, \alpha)$. Therefore, with respect to the new basis, L becomes BA. BA is A with the ith row multiplied by α.

1b. If the vectors were dependent, then we would have a nontrivial relation $a_1w_1 + \cdots + a_i(w_i - \beta w_j) + \cdots + a_mw_m = 0$. Then, $a_1w_1 + \cdots + a_iw_i + \cdots + (a_j - \beta a_i)w_j + \cdots + a_mw_m = 0$. Since the w_k are linearly independent, we must have $a_k = 0$ for $k \neq j$ and $a_j - \beta a_i = 0$. Solving gives $a_j = 0$. However, then our original linear relation is trivial, so our vectors must be independent.

Again, any set of m linearly independent vectors is a basis of an m-dimensional vector space, so the new set of vectors must also be a basis.

To convert a vector $(w_1\ w_2\ \ldots)$ from the original basis to the new basis, we must add βw_j to w_i (in the ith component). $B = E_2(j, i, \beta)$ is the matrix that does this. So, L becomes BA. BA is A with β times the jth row added to the ith row.

1c. We showed in previous homework that the matrix operations of parts (a) and (b) are all that is necessary to row-reduce a matrix. Using the operations of parts (a) and (b) repeatedly to row-reduce A to A', we get the basis $w'_1, \ldots, w'_m$. A' will be the representation of L that will map from $v_1, \ldots, v_n$ to $w'_1, \ldots, w'_m$.

1d. The matrix A' of L with respect to the bases $v_1, \ldots, v_n$ and $w'_1, \ldots, w'_m$ has entries (a_{ij}) given by $L(v_k) = a_{1k}w'_1 + \cdots + a_{mk}w'_m$. Consider i_k where

$L(v_{i_k})$ is not in the span of previous w'_k. Then, $L(v_{i_k})$ maps to w'_k. We see that this means that the i_kth column of A' has one entry of 1 and all other entries of 0 – i.e. it is a pivotal column. Note that the leading one in each row must necessarily be to the right of previous leading ones.

Consider i where $L(v_i)$ is in the span of previous w'_k. Then, $L(v_i)$ is a linear combination of w'_1 through w'_k, so the ith column of A' only has non-zero entries in the first k rows. Combining what we had shown so far, the matrix must be in echelon form.

1e. Essentially, this is the same argument as (1d). For pivotal columns i_j, we must have $L(v_{i_j}) = w'_j$ not in the span of previous $L(v_k)$, i.e. not in the span of previous columns. For non-pivotal columns i, the column is in the span of the previous columns, so we must have $L(v_i)$ in the span of the previous $L(v_k)$.

Thus, the basis must satisfy the properties of the previous construction. In particular, there is only one way that the w'_1 through w'_p could be chosen, since these are determined exactly by the construction.

1f. The matrix representation A' of L is uniquely determined by the choice of bases for V and W. In fact, the representation only depends on w'_1 through w'_p (and not w'_{p+1} through w'_m) because these are the basis of the image of L. Specifically, the vectors w'_{p+1} through w'_m all map to zero regardless of how they are chosen, so they represent rows of zero no matter what. By (e), w'_1 through w'_p are uniquely determined. Moreover, for this basis, the A' of L will be in echelon form. Thus, A' is uniquely determined.

2a and 2b. Note that proving (2b) will also prove (2a). (2b) can be proven directly without using the hint: $\sum_i (-1)^i \dim V_i = \sum_i (-1)^i (\dim \ker d_i + \dim \text{Im } d_i)$. By exactness, this is $-\dim \ker d_1 - \dim \text{Im } d_1 + \sum_{i \geq 2} (-1)^i (\dim \text{Im } d_{i-1} + \dim \text{Im } d_i)$. Reorder the sum as $-\dim \ker d_1 + (-1)^k \dim \text{Im } d_k + \sum_i \dim \text{Im } d_i - \dim \text{Im } d_i = -\dim \ker d_1 + (-1)^k \dim \text{Im } d_k = 0$.

2b. Nevertheless, I would like to give a solution to (2b) using the hint because one needs to understand how to do this. The sequence $0 \to V_1 \to \cdots \to \text{Im } (d_{k-2}) \to 0$ is exact for d_0 through d_{k-3} because it is unchanged from the original exact sequence. The part $V_{k-2} \to \text{Im } (d_{k-2}) \to 0$ is exact because $\text{Im } (d_{k-2})$ is precisely the kernel of the latter map and the image of the former. Next, consider the sequence $0 \to \text{Im } (d_{k-2}) \to V_{k-1} \to V_k \to 0$, where the first map is the embedding (the identity) from $\text{Im } (d_{k-2})$ to V_{k-1}. The exactness follows easily from the fact that $\text{Im } (d_{k-2}) = \ker d_{k-1}$ because the original long exact sequence is exact.

To perform the induction, use a base case of $k = 3$, and the formula holds by part (a). One may also prove it holds for $k = 2$ and $k = 1$ by substituting 0 for V_3 or V_2 and V_3, respectively, and using part (a). Apply the induction hypothesis to the sequence $0 \to V_1 \to \cdots \to \text{Im}\,(d_{k-2}) \to 0$, and use part (a) on $0 \to \text{Im}\,(d_{k-2}) \to V_{k-1} \to V_k \to 0$. Add the resulting equations to get the desired result.