Math 25a Homework 7 Part A Solutions

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1. We have shown in class that the row rank of a matrix is the same as the column rank. That implies that the n columns are linearly independent if and only if the n rows are linearly independent.

A matrix has linearly independent columns if and only if each column is pivotal. Each column is pivotal if and only if it has n pivotal ones when row reduced. A row-reduced n x n matrix can has n pivotal ones if and only if it is the identity. A matrix row reduces to the identity if and only if it is invertible. Therefore, a matrix is invertible if and only if the columns are linearly independent.

2a. First, we need to show the function is defined! From calculus, you know the improper integral $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$. Because $|x^n| \leq 1$ for $x \in [-1, 1]$ and $n \in \mathbb{N}$, the integral $\int_{-1}^{1} \frac{x^n dx}{\sqrt{1-x^2}} = 1$ must also exist. Hence, the integral $\int_{-1}^{1} \frac{p(x)dx}{\sqrt{1-x^2}} = 1$ exists for any polynomial $p(x)$, which shows the integral is defined.

The function is bilinear and symmetric by the additive property of integrals and commutativity of multiplication. $\langle f(x), f(x)\rangle \geq 0$ because $\frac{f(x)^2}{\sqrt{1-x^2}} \geq 0$, so the integral must also be ≥ 0. Moreover, if $f(x) = 0$, then clearly the integral is 0. If the integral $\int_{-1}^{1} \frac{f(x)^2 dx}{\sqrt{1-x^2}}$ is non-zero, then $f(x)$ must be non-zero at some point, so $f(x) \neq 0$. That shows the function is an inner product.

2b. Using the addition formula for cosine, we have $\cos((n+1)a) + \cos((n-1)a) = 2\cos a \cos(na)$. Substituting $a = \cos^{-1} x$ into this formula gives the desired identity.

$T_0(x)$ is 1 and $T_1(x)$ is x. By the way T_{n+1} is defined in terms of T_n and T_{n-1}, we see that T_{n+1} is a polynomial if T_n and T_{n-1} are. Using T_0 and T_1 as the base case, induct to show that T_n is always a polynomial.

2c. We wish to show $\langle T_i(x), T_j(x)\rangle = 0$ whenever $i \neq j$. Make the substitution $x = \cos a$ and evaluate the integral:

$$\int_{-\pi}^{\pi} \frac{\cos(ia)\cos(ja)}{\sin a} \sin a \, da = \int_{-\pi}^{\pi} \cos ia \cos ja \, da$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(i+j)a + \cos(i-j)a \; da$$

Assuming that $i \neq j$, then $i+j$ and $i-j$ are both nonzero. That allows us to integrate to get

$$\frac{\sin(i+j)\pi}{i+j} + \frac{\sin(i-j)\pi}{i-j} - \frac{\sin(i+j)(-\pi)}{i+j} - \frac{\sin(i-j)(-\pi)}{i-j}$$

which is 0, as desired. This shows they are orthogonal.

To show that they form a basis, we just need to show that any polynomial is a linear combination of T_i. We first show that T_i has degree i. Prove this using induction. It holds for $i = 0$ and $i = 1$, and by the relation $T_{i+1} = 2xT_i - T_{i-1}$, we see T_{i+1} has degree one greater than T_i , provided T_{i-1} is of degree less than T_i. That is enough to complete the induction.

Next, given a polynomial P, we inductively show that P is a linear combination of the T_i. We induct on the degree of P. The case $\deg P = 0$ is easy since $T_0 = 1$. Now, assume that all polynomials with degree less than or equal to n are expressible as a linear combination of T_i. Given a polynomial P with $\deg P = n+1$, there is a real number α such that $P - \alpha T_{n+1}$ is of degree n (this is because T_{n+1} is also of degree $n+1$). Then, $P - \alpha T_{n+1}$ is a linear combination of T_i. Therefore, P is also a linear combination of T_i. That completes the proof.

3a. First, we show $L(v)$ is a linear function from V to $\mathbb{R}$. This follows from the bilinearity of the inner product: $L(v)(ax+by) = \langle v, ax+by\rangle = a\langle v,x\rangle + b\langle v,y\rangle = aL(v)(x) + bL(v)(y)$.

Next, we show L is linear. This also follows from the bilinearity of the inner product: given any w, we have $L(ax+by)(w) = \langle ax+by, w\rangle = a\langle x,w\rangle + b\langle y,w\rangle = aL(x)(w) + bL(y)(w)$. Hence, $L(ax+by) = aL(x) + bL(y)$, so L is linear.

3b. Let V have dimension n. V^* is the set of maps from V to $\mathbb{R}$, so it is equivalent to the set of 1xn matrices. Thus, V^* has dimension n. Since V also has dimension n, to show it is an isomorphism, we just have to show it is injective. To show it is injective, we just need to show its kernel is 0. Assume that $L(v)$ is 0. That means $L(v)(w)$ is always 0 for any choice of w. In particular, for $w = v$, we have $L(v)(v) = \langle v, v\rangle = 0$. By the properties of the inner product, $v = 0$. That completes the proof.

Note: When V is infinite dimensional, as it turns out, V^* has greater dimension than V.

3c. We define an inner product on the polynomials by $\langle p,q\rangle = \int_0^1 p(x)q(x)dx$. The proof that this is an inner product is identical to problem (2a) except that there are no denominators to worry about.

Next, observe that the mapping $p \mapsto p(3)$, which maps a polynomial to its value at 3, is linear. This follows from the properties of polynomials:

$(ap + bq)(3) = ap(3) + bq(3)$ for real numbers a, b and polynomials p, q. Let this linear mapping be T.

Note that $T \in P_d^*$. By part (b), we know that given $T \in P_d^*$, then we can find a unique element $p \in P_d$ such that $T(q) = \langle p, q \rangle$. Rewriting this, we have found unique p such that for any $q \in P_d$, we have $\int_0^1 p(x)q(x)dx = q(3)$. That completes the proof.