Problem Set #7 Solutions
Part B
November 22, 2002

(1) (Problem 2.7.4)

(a) To approximate the kth root of a (positive) number a, we will solve the equation $f(x) = x^k - b = 0$ by Newton's method. At the a_n step we have

$$\begin{aligned} a_{n+1} &= a_n - [\mathbf{D}f(a_n)]^{-1} f(a_n) \\ &= a_n - \frac{a_n^k - b}{k a_n^{k-1}} \end{aligned}$$

So $a_{n+1} = g(a_n)$, where $g(a_n) = \frac{(k-1)a_n^k + b}{k a_n^{k-1}}$.

(b) We have $a_{n+1} = \frac{1}{k}\left((k-1)a_n + \frac{b}{a_n^{k-1}}\right)$ is a weighted average between a_n and $\frac{b}{a_n^{k-1}}$.

(2) (Problem 2.7.5 (a)) We will solve $f(x) = x^3 - 9 = 0$ starting with $a_0 = 2$. Using the formula obtained in the previous problem:

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 2 - \frac{(2)^3 - 9}{3(2)^2} = \frac{25}{12} \approx 2.0833333 \\ a_2 &= \frac{25}{12} - \frac{\left(\frac{25}{12}\right)^3 - 9}{3\left(\frac{25}{12}\right)^2} = \frac{23401}{11250} \approx 2.0800888 \\ a_3 &= \frac{19,221,773,312,701}{9,240,864,766,875} \approx 2.0800838 \\ a_4 &\approx 2.0800838 \end{aligned}$$

To six decimal places, then, $9^{\frac{1}{3}} \approx 2.080084$.

(3) (Problem 2.7.14)

(a) If $b > 0$ and $a_0 > 0$ in the expression found in the first problem we see that every term $g(a_n)$ is positive. Now, there are two cases to consider, assuming that the initial guess $a_0 \neq \sqrt[k]{b}$. If $a_0^n < b$, then we obtain a monotone positive increasing sequence bounded above by $\sqrt[k]{b}$, therefore it converges to a limit $a > 0$. Furthermore we see that $\lim_{n\to\infty} a_n = a = \lim_{n\to\infty} \frac{(k-1)a_n^k + b}{k a_n^{k-1}} = \frac{(k-1)a^k + b}{k a^{k-1}}$. Solving for a then gives $a^k = b$; since $a > 0$, we have that a equals the positive kth root of b. Similarly if $a_0^k > b$ we obtain a monotone decreasing sequence bounded below by $\sqrt[k]{b}$, also converging to the positive kth root of b.

(b) A divide and average algorithm will not always work, however. Observe, for instance, the behavior of this method applied to finding the fifth root of 1 when a_n falls between 0.8 and 1 for some n (thanks to William Deringer for this suggestion).

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